Sketching polynomial graphs

How do you sketch the graph of a polynomial without plotting dozens of points, and which parts of the sketch are exact?

WORDS TO KNOW

Factored form: a polynomial written as a product, like $x(x-2)^2(x+3)$.

End behavior: where the two ends of the graph go, set by the leading term.

Test point: a value $f(a)$ worked out to see how high or low the graph is at $x = a$.

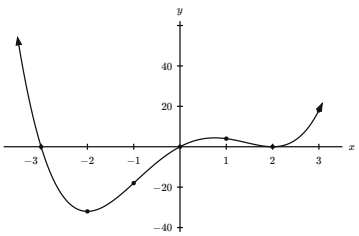
Multiplicity: how many times a factor repeats. Odd: the graph crosses. Even: it touches and turns back.

Sign chart: a table of where the polynomial is positive (above the axis) or negative (below).

Turning point: a peak or a valley.

KEY IDEAS

1. A sketch from factored form, piece by piece



1. **Zeros:** each linear factor gives one. Odd multiplicity crosses, even touches.
2. **Ends:** multiply the leading terms of the factors to get the leading term. No expanding needed.
3. **y-intercept:** $f(0)$.
4. **Signs:** the graph has no breaks, so it can only change sign at a zero. Find the sign on each interval between zeros.
5. **Test points:** a value or two in each interval shows how far up or down the graph goes.
6. **Turns:** between two neighboring zeros there is at least one peak or valley. A degree n polynomial has at most $n-1$ turns.

2. Factor first

An expanded rule hides its zeros, so factor it completely first: take out a common factor, look for a quadratic in x^2, group four terms, or divide out a known zero.

A quadratic factor with no real zeros, like x^2+1, has one sign for every x. It adds no zero and no crossing, but it still counts toward the degree and the leading term.

3. What a sketch knows exactly

The zeros, the y-intercept, the signs and the ends are exact. A peak or valley between zeros is not: test points only bracket it. A touching zero is an exact peak or valley, because it sits on the axis.

WATCH OUT

- **Check every quadratic factor.** x^2-9 hides two zeros; x^2+9 has none.
- **A flat crossing is not a turn.** At a zero of multiplicity 3 the graph levels off but keeps going the same way.
- **Do not place peaks exactly.** The highest test point is only near the peak.
- **In a model, only the domain counts.** Draw the graph only where the situation makes sense.

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WORKED EXAMPLES

Example 1. Sketch the graph of $f(x) = (x+2)(x-1)^2(x-5)$.

1. **Zeros:** -2 and 5 are simple, so the graph crosses there. 1 is a double zero, so the graph touches.
2. **Ends:** the leading term is $x \cdot x^2 \cdot x = x^4$. Both ends rise.
3. **y-intercept:** $f(0) = (2)(1)(-5) = -10$.
4. **Signs:** for $x > 5$ every factor is positive, so $f > 0$. Moving left, the sign flips at 5, stays the same at 1 (even), and flips at -2: so $+$ for $x < -2$, $-$ on $-2 < x < 1$, $-$ on $1 < x < 5$, $+$ for $x > 5$.
5. **Test points:** $f(-1) = (1)(4)(-6) = -24$ and $f(3) = (5)(4)(-2) = -40$.
6. **Turns:** a valley between -2 and 1 (at least 24 deep), a peak on the axis at $(1, 0)$, where the graph touches from below, and a valley between 1 and 5 (at least 40 deep). That is 3 turns, the most degree 4 allows.

Answer: from the top left, the graph comes down through -2, dips to a valley, rises to touch the axis at $(1, 0)$, dips to a second valley, crosses at 5 and rises. **Check:** $f(6) = (8)(25)(1) = 200$, positive, as the sign chart says.

Example 2. Sketch the graph of $h(x) = x^3 + 2x^2 - 8x$.

1. Every term has x: $h(x) = x(x^2 + 2x - 8)$. Two numbers with product -8 and sum 2 are 4 and -2, so $h(x) = x(x+4)(x-2)$.
2. The zeros -4, 0 and 2 are all simple. The leading term is x^3: the left end falls and the right end rises. $h(0) = 0$.
3. Signs, from the right: $+$ for $x > 2$, $-$ on $0 < x < 2$, $+$ on $-4 < x < 0$, $-$ for $x < -4$.
4. Test points: $h(-2) = -8 + 8 + 16 = 16$ and $h(1) = 1 + 2 - 8 = -5$.

Answer: the graph rises through -4 to a peak (at least 16 high), comes down through the origin to a valley (at least 5 deep), and rises through 2. Two turns, the most a cubic can have. **Check:** $x(x+4)(x-2)$ multiplies out to $x^3 + 2x^2 - 8x$.

Example 3. Sketch the graph of $m(x) = x^3 - x^2 + 3x - 3$.

1. Group in pairs: $x^2(x-1) + 3(x-1) = (x^2+3)(x-1)$.
2. Since $x^2 + 3 \geq 3$, it is never zero and always positive. The only zero is 1, and it is simple.
3. So m has the sign of $x - 1$: negative for $x < 1$, positive for $x > 1$. The leading term is x^3, so the left end falls and the right end rises. $m(0) = (3)(-1) = -3$.
4. Test points: $m(-1) = (4)(-2) = -8$, $m(2) = (7)(1) = 7$ and $m(3) = (12)(2) = 24$. They keep rising.
5. Rising test points cannot rule out a tiny peak and valley between them: $x^3 - 2x^2 + x - 2 = (x^2+1)(x-2)$ has a pair too small to see. For a cubic, the test $b^2 - 3ac$ settles it: $(-1)^2 - 3(1)(3) = -8 < 0$, so m never turns.

Answer: one crossing, going up at $(1, 0)$, through $(0, -3)$, and the graph rises the whole way. **Check:** $(x^2+3)(x-1) = x^3 - x^2 + 3x - 3$.

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