Graphing rational functions

How do you sketch the graph of a rational function by hand, with its asymptotes, holes, intercepts and the right shape on every piece?

WORDS TO KNOW

Lowest terms: the formula after every common factor of top and bottom is canceled.

Multiplicity: how many times a factor appears. $(x-2)^2$ has multiplicity 2, an even number.

Slant line: a slanted line the graph follows far out, when the top's degree is one more than the bottom's.

Hole: a single missing point, where a factor canceled completely. Drawn as an open circle.

Crossing / touch: at a zero of odd multiplicity the graph crosses the axis; at even multiplicity it touches and turns back.

KEY IDEAS

1. The steps of a sketch

1. **Factor** the top and the bottom. **Cancel** common factors. Each factor that cancels completely leaves a **hole**, at the height the canceled formula gives.
2. **Vertical asymptotes:** the real zeros of the lowest-terms denominator. Find C (put $x = a$ into every other factor) to see which way the graph goes on each side.
3. **Far out:** compare degrees. Lower on top gives $y = 0$; equal gives $y = \frac{a}{b}$; one higher gives a slant line found by long division.
4. **Intercepts:** zeros of the lowest-terms numerator; the y-intercept is $f(0)$, if 0 is in the domain.
5. **Signs:** the zeros and vertical asymptotes cut the axis into intervals. One test value fixes the sign of each whole interval.
6. **Crossings:** solve $f(x) = L$ to see where the graph meets its horizontal asymptote.
7. **Join:** draw one branch between each pair of neighboring vertical asymptotes, through the points found.

2. Reading signs without a test value

Far to the right, the sign of f is the sign of the ratio of the leading coefficients. Move left across the breaks:

- At an **odd** power (a zero or an asymptote), the sign **flips**.
- At an **even** power, the sign **stays**.
- A hole never changes the sign.

WATCH OUT

- **Cancel completely before you apply the rules.** A zero of the original denominator may be a hole.
- **A hole is never a zero,** even when it sits on the axis. The function has no value there.
- **The graph may cross its horizontal asymptote,** but it never meets a vertical one.
- **A sketch fixes the shape, not exact turning points.** Finding the exact top of a hump needs calculus.

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WORKED EXAMPLES

Example 1. Sketch the graph of $f(x) = \dfrac{x^2 - 4x - 5}{x^2 + x - 6}$.

1. Factor: $f(x) = \dfrac{(x-5)(x+1)}{(x+3)(x-2)}$. Nothing cancels, so there is no hole.
2. Vertical asymptotes $x = -3$ and $x = 2$, both odd. Near -3: $C = \dfrac{(-8)(-2)}{-5} = -\dfrac{16}{5}$, so the graph goes up on the left and down on the right. Near 2: $C = \dfrac{(-3)(3)}{5} = -\dfrac{9}{5}$, so again up on the left and down on the right.
3. Equal degrees, leading coefficients 1 and 1: horizontal asymptote $y = 1$.
4. Zeros 5 and -1, both crossings. y-intercept $f(0) = \dfrac{-5}{-6} = \dfrac{5}{6}$.
5. Signs: far right is positive. Every break is odd, so moving left the signs alternate: positive on $(5, \infty)$, negative on $(2, 5)$, positive on $(-1, 2)$, negative on $(-3, -1)$, positive on $(-\infty, -3)$.
6. Crossing: $x^2 - 4x - 5 = x^2 + x - 6$ gives $-5x = -1$, so the graph meets $y = 1$ at $\left(\dfrac{1}{5}, 1\right)$.
7. Join: the left branch comes in above $y = 1$ and climbs along $x = -3$. The middle branch rises from the bottom, crosses the axis at -1, passes $\left(0, \dfrac{5}{6}\right)$, crosses $y = 1$ and climbs along $x = 2$. The right branch rises from the bottom, crosses at 5, and settles toward $y = 1$ from below.

Answer: three branches, with the features above. **Check:** $f(3) = \dfrac{(-2)(4)}{(6)(1)} = -\dfrac{4}{3}$, negative on $(2, 5)$, and $f(10) = \dfrac{55}{104}$, below 1.

Example 2. Sketch the graph of $g(x) = \dfrac{x^2 - 4x}{x^3 + 2x^2 + x}$.

1. Factor: $g(x) = \dfrac{x(x-4)}{x(x+1)^2}$. Cancel x: $g(x) = \dfrac{x-4}{(x+1)^2}$ for $x \neq 0$. The hole is at $\left(0, \dfrac{0-4}{1}\right) = (0, -4)$.
2. Vertical asymptote $x = -1$, an even power, with $C = -1 - 4 = -5$. Negative over a square: down on both sides.
3. Lowest terms: degree 1 over degree 2, so the horizontal asymptote is $y = 0$.
4. Zero $x = 4$, a crossing. No y-intercept, because 0 is not in the domain.
5. Signs: far right is positive. Moving left, the sign flips at 4 (odd) and stays at -1 (even): positive on $(4, \infty)$, negative on $(-1, 4)$ and on $(-\infty, -1)$.
6. Join: the left branch runs just under the axis and falls along $x = -1$. The right branch rises from the bottom, passes the open circle at $(0, -4)$, crosses the axis at 4, then settles toward the axis from above.

Answer: a hole at $(0, -4)$, vertical asymptote $x = -1$ (down on both sides), horizontal asymptote $y = 0$, one zero at 4. **Check:** $g(-2) = \dfrac{-6}{1} = -6$ and $g(1) = \dfrac{-3}{4}$, both negative.

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