Polynomial and rational inequalities

Where is a polynomial, or a fraction of polynomials, positive or negative? Find the few points where the sign can change, then test each stretch between them.

WORDS TO KNOW

Critical numbers: the real zeros of the expression, and for a fraction also the zeros of its denominator. Only there can the sign change.

Test number: one input picked inside a stretch to find the sign there.

Interval notation: writing a solution set as intervals such as $(-2, 5]$, joined with $\cup$.

Sign chart: a number line split at the critical numbers, with the sign of the expression marked on each stretch.

Multiplicity: the exponent k of a factor $(x - r)^k$.

KEY IDEAS

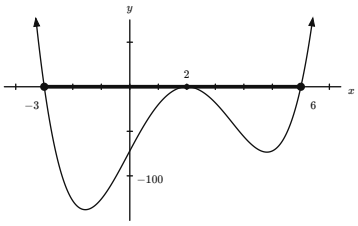
1. Signs change only at zeros and breaks

The graph of a polynomial is one unbroken curve, so it can change sign only at its real zeros. A rational function can also jump across a vertical asymptote, where its denominator is 0. Between two neighboring critical numbers the sign stays fixed, so one test number speaks for the whole stretch.

1. **Compare with zero.** Move every term to one side. For fractions, combine into one fraction $\frac{N(x)}{D(x)}$.
2. **Factor completely** and list the critical numbers in order.
3. **Find the sign** on each stretch, with a test number or from the factors.
4. **Decide the endpoints:** zeros of the expression are in only for $\leq$ or $\geq$. Zeros of the denominator are never in.
5. **Write the answer** in interval notation.

2. Read the signs from the factors

Far to the right, every factor $x - r$ is positive, so the sign there is the sign of the leading coefficients. Moving left, the sign flips at a critical number of odd multiplicity and stays the same at one of even multiplicity, because an even power is never negative. On the graph, the curve crosses the axis at an odd zero and only touches it at an even zero.



A factor with no real zeros, such as $x^2 + 7$, is positive for every x. It never changes the sign and adds no critical number.

WATCH OUT

- **Do not multiply or divide both sides by an expression in x whose sign you do not know.** A factor like $x - 5$ is negative for some x and positive for others, so the inequality would need to flip on only part of the line. Subtract to get 0 on one side instead.
- **A zero of the denominator is never in the answer**, even with $\leq$ or $\geq$. The expression has no value there.
- **Do not drop a zero of even multiplicity under $\leq$ or $\geq$.** It can be a single point in the answer, all on its own.
- **Find every real zero.** A missed zero means a missed sign change.

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WORKED EXAMPLES

Example 1. Solve $(x+4)(x-1)^2(x-3) \geq 0$.

1. It is already factored and compared with 0. The critical numbers are −4, 1 and 3.
2. The leading coefficient is positive, so the sign is $+$ on $(3, \infty)$.
3. Moving left: 3 has multiplicity 1, so the sign flips to $-$ on $(1, 3)$. Then 1 has multiplicity 2, so it stays $-$ on $(-4, 1)$. Then −4 flips it to $+$ on $(-\infty, -4)$.
4. **Check** with test numbers: $x = 2$ gives $6 \cdot 1 \cdot (-1) = -6$, and $x = 0$ gives $4 \cdot 1 \cdot (-3) = -12$.
5. The inequality allows 0, so all three zeros are in. The zero at 1 sits between two negative stretches, so it is in on its own.

Answer: $(-\infty, -4] \cup \{1\} \cup [3, \infty)$.

Example 2. Solve $\dfrac{2x-1}{x+3} \leq 1$.

1. Do not multiply by $x+3$: its sign is unknown. Subtract 1 over the common denominator:
$$\frac{2x-1}{x+3} - \frac{x+3}{x+3} = \frac{x-4}{x+3} \leq 0$$
2. Critical numbers: 4 (zero of the top) and −3 (zero of the bottom).
3. Signs: $+$ on $(4, \infty)$, $-$ on $(-3, 4)$, $+$ on $(-\infty, -3)$.
4. The zero 4 is in, because of $\leq$. The break −3 is out.

Answer: $(-3, 4]$. **Check:** $x = 0$ gives $-\frac{1}{3} \leq 1$, true. $x = 5$ gives $\frac{9}{8} \leq 1$, false.

Example 3. Solve $\dfrac{x^2-1}{x^2-2x-8} \geq 0$.

1. Factor the top and the bottom. No factor cancels.
$$\frac{x^2-1}{x^2-2x-8} = \frac{(x+1)(x-1)}{(x+2)(x-4)}$$
2. Critical numbers in order: −2, −1, 1, 4, each of multiplicity 1.
3. The leading coefficients are both 1, so the sign is $+$ on $(4, \infty)$ and flips at each critical number: $+, -, +, -, +$ from left to right.
4. The zeros of the top, −1 and 1, are in. The zeros of the bottom, −2 and 4, are out.

Answer: $(-\infty, -2) \cup [-1, 1] \cup (4, \infty)$. **Check:** $x = 0$ gives $\frac{-1}{-8} = \frac{1}{8} \geq 0$, true.

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