Slant asymptotes

When the top of a rational function has a higher degree than the bottom, the graph has no horizontal asymptote. Far out, can it still settle along a straight line?

WORDS TO KNOW

Slant asymptote (or oblique asymptote): a line $y = mx + b$ with $m \neq 0$ that the graph gets closer and closer to far out, as $x \to \infty$ or $x \to -\infty$.

Degree: the highest power of x in a polynomial.

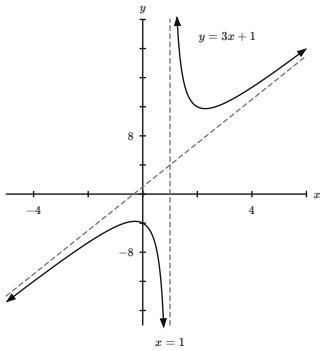
Quotient and remainder: long division of $P(x)$ by $Q(x)$ gives $P(x) = Q(x) \cdot (\text{quotient}) + R(x)$, where the remainder R has lower degree than Q.

Gap: the vertical distance $f(x) - (mx + b)$ from the line to the graph.

KEY IDEAS

1. Divide, and the quotient is the line

A rational function $f(x) = \frac{P(x)}{Q(x)}$ has a slant asymptote exactly when the degree of P is one more than the degree of Q and the division does not come out even. Far out, the leftover fraction is tiny, because its bottom has the higher degree. What is left to see is the line.



Divide: $f(x) = \underbrace{mx + b}_{\text{quotient}} + \dfrac{R(x)}{Q(x)}$

The slant asymptote is $y = mx + b$. Its slope m is the leading coefficient of P divided by the leading coefficient of Q.

2. The leftover fraction says where the graph is

The graph is **above** the line where $\frac{R(x)}{Q(x)} > 0$, and **below** it where $\frac{R(x)}{Q(x)} < 0$.

The graph **meets** the line exactly where $R(x) = 0$ and $Q(x) \neq 0$.

If Q is linear, the remainder is a nonzero number, so the graph never meets its slant line. If Q has degree 2 or more, the graph can cross it. An asymptote only describes the ends of the graph.

3. Which far-out shape?

Degree of top compared with bottom	Far out, the graph follows
lower or equal	a horizontal line
one more	a slant line
two or more higher	a curve (the quotient), no line

A rational function has at most one horizontal or slant asymptote, and it is the same line on both ends.

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WATCH OUT

- **Do not just divide the leading terms.** That gives the slope only. Finish the division to get the whole line, $y = mx + b$.
- **Use the leading coefficient of the bottom.** For a bottom like $4x + 3$, the first step divides by $4x$, not by x.
- **A zero remainder means no asymptote.** Then f is the quotient line itself with a hole, like $\frac{x^2-25}{x+5} = x - 5$ for $x \neq -5$.
- **Crossing is allowed.** "Asymptote" promises that the gap shrinks far out, not that the graph never touches the line.

WORKED EXAMPLES

Example 1. Find the slant and vertical asymptotes of $g(x) = \dfrac{-6x^2 + 5x + 3}{3x - 1}$, and the side of the slant line the graph is on at each end.

1. Degrees 2 and 1, so there is a slant asymptote. Divide: $-6x^2 \div 3x = -2x$, and $-2x(3x - 1) = -6x^2 + 2x$ leaves $3x + 3$.
2. Next, $3x \div 3x = 1$, and $1 \cdot (3x - 1)$ leaves 4. So
$$g(x) = -2x + 1 + \frac{4}{3x - 1}$$
3. Far right, $3x - 1 > 0$, so the gap is positive: the graph is above the line. Far left, $3x - 1 < 0$: below.
4. The bottom is 0 at $x = \frac{1}{3}$, where the top is $-\frac{2}{3} + \frac{5}{3} + 3 = 4 \neq 0$: a vertical asymptote.

Answer: slant asymptote $y = -2x + 1$, vertical asymptote $x = \frac{1}{3}$; above the line on the right, below it on the left. **Check:** $(3x - 1)(-2x + 1) + 4 = -6x^2 + 5x - 1 + 4$, the top.

Example 2. Find the slant asymptote of $k(x) = \dfrac{x^3 + 3x^2 - x + 1}{x^2 + 1}$ and where the graph meets it.

1. Divide: $x(x^2 + 1) = x^3 + x$ leaves $3x^2 - 2x + 1$. Then $3(x^2 + 1)$ leaves $-2x - 2$. So
$$k(x) = x + 3 - \frac{2x + 2}{x^2 + 1}$$
2. The slant asymptote is $y = x + 3$. The bottom $x^2 + 1$ is never 0, so there is no vertical asymptote.
3. The remainder $-2x - 2$ is 0 at $x = -1$. There the line gives $y = -1 + 3 = 2$.
4. The bottom is always positive, so the side depends on $-(2x + 2)$: above the line for $x < -1$, below it for $x > -1$.

Answer: $y = x + 3$; the graph crosses it once, at (−1, 2). **Check:** $k(-1) = \frac{-1+3+1+1}{2} = 2$.

Example 3. Does $p(x) = \dfrac{x^3 - 2}{x + 1}$ have a slant asymptote?

1. The degrees are 3 and 1, two apart. Division gives $x^3 - 2 = (x + 1)(x^2 - x + 1) - 3$.
2. So the quotient is $x^2 - x + 1$ and the leftover fraction is $-\frac{3}{x+1}$. The fraction shrinks far out, but the quotient is not a line.

Answer: no. Far out the graph follows the parabola $y = x^2 - x + 1$ instead.

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