Precalculus Unit 1 quiz 1: Degree, end behavior and the shape of a polynomial graph, Graphing polynomial functions, Graphing rational functions (answer key)

7 questions, 20 points

ANSWERS AT A GLANCE

1. $+$ on $(-\infty, -2)$, $-$ on $(-2, 1)$, $+$ on $(1, 3)$ and $(3, \infty)$. The graph crosses the axis at $x = -2$ and touches it at $x = 3$; it goes down on the left of $x = 1$ and up on the right.
2. (a) degree 4, leading coefficient 8 (b) as $x \to \pm\infty$, $p(x) \to \infty$ (c) at most 4 real zeros and at most 3 turning points
3. (a) k^3x^6; every $k < 0$ (b) degree 3, leading term $-8x^3$; as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$
4. Left end falling and right end rising; one crossing of the x-axis; a peak followed by a valley, both on the same side of the x-axis as each other and away from the zero (both above the axis with the zero to their left, or both below with the zero to their right). The range is all real numbers.
5. (a) $f(x) = (x-3)^2(x+1)^2$; -1 and 3, each of multiplicity 2 (b) $f(x)$ is a square; as $x \to \pm\infty$, $f(x) \to \infty$; y-intercept $(0, 9)$ (c) valleys $(-1, 0)$ and $(3, 0)$, peak $(1, 16)$.
   Features to grade on the sketch: touches the axis at $(-1, 0)$ and $(3, 0)$ and never goes below it; both ends up; through $(0, 9)$; a peak at $(1, 16)$; three turns.
6. (a) m even, n even. (b) $-$ on $(-\infty, -1)$, $(-1, 2)$ and $(2, 4)$, $+$ on $(4, \infty)$. (c) Every even $n \geq 4$; no, never $y = 1$.
7. Zeros -1 and 2, both simple (the factor $x^2 + 2$ has none); as $x \to \pm\infty$, $f(x) \to -\infty$; y-intercept $(0, 4)$; negative for $x < -1$, positive for $-1 < x < 2$, negative for $x > 2$; $f(1) = 6$; one turning point, a peak in $-1 < x < 2$.
   Features to grade on the sketch: crossings at $(-1, 0)$ and $(2, 0)$ only; both ends down; through $(0, 4)$ and $(1, 6)$; one peak, at least 6 high, between -1 and 2.

WORKED SOLUTIONS

1. Let $f(x) = \dfrac{(x-3)^2(x+2)}{x-1}$. Give the sign of f on each interval. Then use the signs to say whether the graph crosses or touches the axis at each zero, and what it does on each side of $x = 1$. (2 points)

> **ANSWER**
> $+$ on $(-\infty, -2)$, $-$ on $(-2, 1)$, $+$ on $(1, 3)$ and $(3, \infty)$. The graph crosses the axis at $x = -2$ and touches it at $x = 3$; it goes down on the left of $x = 1$ and up on the right.
>
> **SOLUTION**
> Nothing cancels: $x - 1$ is not a factor of the numerator, which is (4)(3) = 12 ≠ 0 at $x = 1$. The breaks are -2 (power 1), 1 (the denominator, power 1) and 3 (power 2). Far right the leading coefficients are 1 and 1, so $+$ on $(3, \infty)$. Stay at 3 (even): $+$ on $(1, 3)$. Flip at 1: $-$ on $(-2, 1)$. Flip at -2: $+$ on $(-\infty, -2)$. Check: $f(0) = \dfrac{18}{-1} = -18 < 0$ and $f(2) = \dfrac{4}{1} = 4 > 0$.
> The sign changes at -2, so the graph crosses the axis there; it is $+$ on both sides of 3, so the graph touches the axis at $(3, 0)$ and turns back. At $x = 1$ the denominator is 0 and the numerator is not, so $x = 1$ is a vertical asymptote: f is negative just left of it and positive just right, so $f(x) \to -\infty$ as $x \to 1^-$ and $f(x) \to \infty$ as $x \to 1^+$.
>
> **SCORING**
> 1 point: all four signs. 1 point: the crossing at -2, the touch at 3, and down on the left and up on the right of $x = 1$, each read from the signs.

2. Let $p(x) = 2x^4(x+4) - 2x^5 - 6x + 7$. (3 points)

(a) Find the degree and the leading coefficient of p.

(b) Describe the end behavior of p in arrow notation.

(c) What is the greatest number of real zeros p can have, and the greatest number of turning points?

> **ANSWER**

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(a) degree 4, leading coefficient 8 (b) as $x \to \pm\infty$, $p(x) \to \infty$ (c) at most 4 real zeros and at most 3 turning points

SOLUTION

(a) Distribute first: $2x^4(x+4) = 2x^5 + 8x^4$. The x^5 terms cancel, so

$$p(x) = 8x^4 - 6x + 7,$$

of degree 4 with leading coefficient 8. (The degree is not 5: the terms that would give it add to 0.)

(b) The degree is even and the leading coefficient is positive, so as $x \to \pm\infty$, $p(x) \to \infty$.

(c) A polynomial of degree 4 has at most 4 real zeros and at most 4 − 1 = 3 turning points.

SCORING

1 point: degree 4 and leading coefficient 8, after the x^5 terms cancel (no credit for degree 5). 1 point: correct end behavior in arrow notation for both ends (follow through from a wrong (a)). 1 point: both bounds, 4 zeros and 3 turning points.

3. Let $p(x) = (kx-2)^3(x+1)(3-x)^2$, where k is a number. (3 points)

(a) Suppose $k \neq 0$. Find the leading term of p in terms of k, and find every value of k for which both ends of the graph fall.

(b) Suppose $k = 0$. Find the degree and the leading term of p, and describe its end behavior in arrow notation.

ANSWER

(a) k^3x^6; every $k < 0$ (b) degree 3, leading term $-8x^3$; as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$

SOLUTION

(a) The leading term of a product is the product of the factors' leading terms: $(kx)^3 \cdot x \cdot (-x)^2 = k^3x^3 \cdot x \cdot x^2 = k^3x^6$. The degree 6 is even, so both ends fall exactly when the leading coefficient k^3 is negative, that is when $k < 0$.

(b) With $k = 0$ the first factor is the constant $(-2)^3 = -8$, so $p(x) = -8(x+1)(3-x)^2$. Its leading term is $-8 \cdot x \cdot x^2 = -8x^3$: degree 3, not 6. Odd degree, negative leading coefficient: as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$.

SCORING

1 point: leading term k^3x^6. 1 point: $k < 0$, with the reason that even degree needs a negative leading coefficient for both ends to fall. 1 point: for $k = 0$, degree 3, leading term $-8x^3$ and the correct end behavior.

4. Sketch the graph of a polynomial with all of these features:

- degree 5 and a positive leading coefficient,
- exactly one real zero,
- exactly two turning points.

Then give the range of your polynomial. (3 points)

ANSWER

Left end falling and right end rising; one crossing of the x-axis; a peak followed by a valley, both on the same side of the x-axis as each other and away from the zero (both above the axis with the zero to their left, or both below with the zero to their right). The range is all real numbers.

SOLUTION

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Odd degree and a positive leading coefficient: the left end falls and the right end rises. Two turning points with a rising left part means the graph rises to a peak, falls to a valley, then rises for good.

For only one zero, the peak and the valley must be on the same side of the axis. If the zero comes before the peak, the graph is above the axis from there on, so the valley stays above the axis. If the zero comes after the valley, the peak is below the axis. A peak above the axis and a valley below it would give three crossings.

One such polynomial is $p(x) = x^5 - 5x + 10$: its peak is $(-1, 14)$, its valley is $(1, 6)$, and it crosses the axis once, between -2 and -1 (since $p(-2) = -12$ and $p(-1) = 14$).

The degree is odd, so the range is all real numbers.

SCORING

1 point: left end falls and right end rises. 1 point: exactly one crossing and exactly two turns, with the peak and the valley on the same side of the axis. 1 point: range all real numbers.

5. Let $f(x) = (x^2 - 2x - 3)^2$. (3 points)

(a) Write $f(x)$ as a product of linear factors, and give each zero with its multiplicity.

(b) Explain why $f(x) \geq 0$ for every x. Give the end behavior and the y-intercept, and sketch the graph.

(c) Find every peak and valley of the graph exactly.

ANSWER

(a) $f(x) = (x - 3)^2(x + 1)^2$; -1 and 3, each of multiplicity 2 (b) $f(x)$ is a square; as $x \to \pm\infty$, $f(x) \to \infty$; y-intercept $(0, 9)$ (c) valleys $(-1, 0)$ and $(3, 0)$, peak $(1, 16)$.

Features to grade on the sketch: touches the axis at $(-1, 0)$ and $(3, 0)$ and never goes below it; both ends up; through $(0, 9)$; a peak at $(1, 16)$; three turns.

SOLUTION

(a) $x^2 - 2x - 3 = (x - 3)(x + 1)$, so $f(x) = (x - 3)^2(x + 1)^2$. The zeros -1 and 3 each have multiplicity 2.

(b) $f(x)$ is the square of a real number for every x, so $f(x) \geq 0$, and it is 0 only at the two zeros. The leading term is $(x^2)^2 = x^4$, so both ends rise. $f(0) = (-3)^2 = 9$. The graph comes down from the top left, touches the axis at $(-1, 0)$, rises through $(0, 9)$ to a peak, comes down to touch the axis at $(3, 0)$, and rises off to the top right.

(c) The two touching zeros are the valleys, exactly: $(-1, 0)$ and $(3, 0)$. For the peak, complete the square inside: $x^2 - 2x - 3 = (x - 1)^2 - 4$. For $-1 \leq x \leq 3$, $(x - 1)^2$ is between 0 and 4, so $x^2 - 2x - 3$ is between -4 and 0, and its square is largest, 16, when $x = 1$. The peak is $(1, 16)$. Three turns is $n - 1$ for degree 4, so there are no others.

SCORING

1 point: $(x - 3)^2(x + 1)^2$, with both zeros of multiplicity 2. 1 point: the reason $f(x) \geq 0$, both ends up, y-intercept $(0, 9)$, and a sketch that touches at -1 and 3. 1 point: the peak $(1, 16)$ found exactly (completing the square or an equivalent argument), and the valleys $(-1, 0)$ and $(3, 0)$.

6. Let $f(x) = \dfrac{(x + 1)^m(x - 4)}{(x - 2)^n}$, where m and n are positive whole numbers. Its graph touches the horizontal axis at $x = -1$ without crossing, and goes down on both sides of $x = 2$. (3 points)

(a) Is m even or odd? Is n even or odd? Explain.

(b) Give the sign of f on each interval.

(c) Take $m = 2$. For which n does the graph have the horizontal asymptote $y = 0$? Can it have the horizontal asymptote $y = 1$?

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ANSWER

(a) m even, n even. (b) $-$ on $(-\infty, -1)$, $(-1, 2)$ and $(2, 4)$, $+$ on $(4, \infty)$. (c) Every even $n \geq 4$; no, never $y = 1$.

SOLUTION

(a) A touch without crossing means the factor $x + 1$ appears an even number of times, so m is even. The same direction on both sides of $x = 2$ means $x - 2$ appears an even number of times below, so n is even. (Check the direction: at $x = 2$ the other factors give $C = 3^m \cdot (-2) < 0$, so "down on both sides" fits.)

(b) With m and n even, $(x + 1)^m \geq 0$ and $(x - 2)^n > 0$ wherever f is defined, so f has the sign of $x - 4$, except that it is 0 at $x = -1$: $-$ on $(-\infty, -1)$, $(-1, 2)$ and $(2, 4)$, and $+$ on $(4, \infty)$. (From the far right: $+$, a flip at the odd break 4, no flip at the even breaks 2 and −1.)

(c) For $m = 2$ the numerator has degree 3 and the denominator degree n, and nothing cancels. The horizontal asymptote is $y = 0$ exactly when $3 < n$; with n even that is $n = 4, 6, 8, \ldots$. The asymptote would be $y = 1$ (leading coefficients 1 and 1) only if the degrees were equal, $n = 3$, but n must be even. So never $y = 1$. (For $n = 2$ the top is one degree higher and the graph follows a slant line.)

SCORING

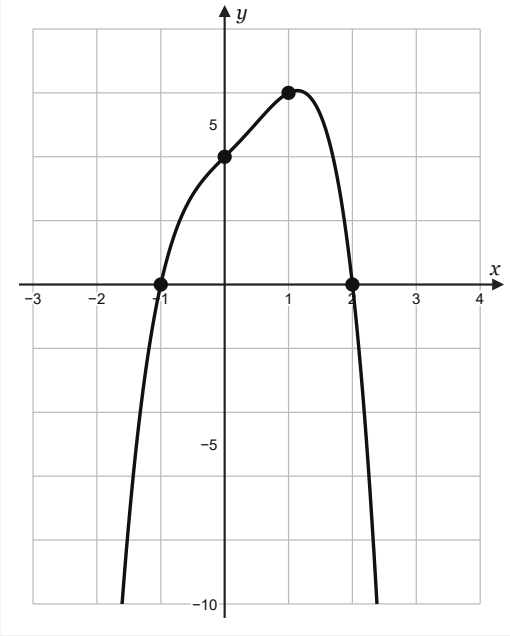
1 point: m and n both even, each with its reason (touch; same direction on both sides). 1 point: all four signs. 1 point: every even $n \geq 4$, and no $y = 1$ because equal degrees would need $n = 3$, which is odd.

7. Sketch the graph of $f(x) = -(x^2 + 2)(x + 1)(x - 2)$. Give its zeros, its end behavior, its y-intercept, the sign of $f(x)$ on each interval and the value $f(1)$, and say how many turning points your sketch has. (3 points)

ANSWER

Zeros −1 and 2, both simple (the factor $x^2 + 2$ has none); as $x \to \pm\infty$, $f(x) \to -\infty$; y-intercept (0, 4); negative for $x < -1$, positive for $-1 < x < 2$, negative for $x > 2$; $f(1) = 6$; one turning point, a peak in $-1 < x < 2$.

Features to grade on the sketch: crossings at (−1, 0) and (2, 0) only; both ends down; through (0, 4) and (1, 6); one peak, at least 6 high, between −1 and 2.



SOLUTION

Since $x^2 + 2 \geq 2$, that factor is never 0 and always positive: it adds no zero and no sign change. The zeros are −1 and 2, both simple, so the graph crosses at each.

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The leading term is $-1 \cdot x^2 \cdot x \cdot x = -x^4$: even degree, negative coefficient, so both ends fall. $f(0) = -(2)(1)(-2) = 4$.

The sign of f is the sign of $-(x+1)(x-2)$: negative for $x < -1$, positive for $-1 < x < 2$, negative for $x > 2$.

$f(1) = -(3)(2)(-1) = 6$. The graph rises from the bottom left, crosses at -1, climbs through $(0, 4)$ to a peak at least 6 high, comes down through 2 and falls off to the bottom right. One turn.

SCORING

1 point: zeros -1 and 2 only, with the reason that $x^2 + 2$ is never 0. 1 point: both ends down, y-intercept $(0, 4)$ and the sign on each interval. 1 point: a sketch with exactly these crossings, through $(0, 4)$ and $(1, 6)$, and one peak.

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