Precalculus Unit 1 quiz 2: The difference quotient, Rates at an instant from shrinking intervals, Concavity: when the rate of change itself changes (answer key)

8 questions, 19 points

ANSWERS AT A GLANCE

1. $-6x - 3h + 24$
2. $\dfrac{1}{\sqrt{x+h}+\sqrt{x}} - 2$, for $x \geq 0$ and $x + h \geq 0$, not both 0.
3. $y = -2x - 1$; $f(3.1) \approx -7.2$
4. (a) $\dfrac{4 - x^2 - xh}{(x^2+4)((x+h)^2+4)}$; every x and $x + h$ are allowed (only $h \neq 0$), since $x^2 + 4$ is never 0. (b) $\dfrac{1}{8}$
5. (a) For $h > 0$, $4 + h$ is not in the domain. (b) -10 and -100. (c) No: only the left side exists, and the left estimates grow without bound and approach no number.
6. 9, 7, 5, 4 cm per day; increasing and concave down (the sunflower grows by less each step).
7. $p = 4$; f is concave up.
8. (a) 2.7 meters per second. (b) 15.0 meters. (c) 3.0 meters per second; the centered average is the best estimate.

WORKED SOLUTIONS

1. Find and simplify the difference quotient of $f(x) = -3(x-4)^2 - 7$. (2 points)

ANSWER

$-6x - 3h + 24$

SOLUTION

$f(x+h) = -3(x-4+h)^2 - 7$. Treat $x - 4$ as one quantity: $(x-4+h)^2 = (x-4)^2 + 2(x-4)h + h^2$. Subtracting $f(x)$ removes $-3(x-4)^2$ and the constant -7:

$$f(x+h) - f(x) = -3\left[2(x-4)h + h^2\right].$$

Every term carries h; cancel it, since $h \neq 0$:

$$\frac{f(x+h) - f(x)}{h} = -6(x-4) - 3h = -6x - 3h + 24.$$

SCORING

1 point for $f(x+h) - f(x) = -3\left[2(x-4)h + h^2\right]$ or its expanded form; 1 point for $-6x - 3h + 24$.

2. Find and simplify the difference quotient of $f(x) = \sqrt{x} - 2x$. (2 points)

ANSWER

$\dfrac{1}{\sqrt{x+h}+\sqrt{x}} - 2$, for $x \geq 0$ and $x + h \geq 0$, not both 0.

SOLUTION

Split the quotient into its two parts:

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} - \frac{2(x+h) - 2x}{h}.$$

The second part is $\dfrac{2h}{h} = 2$. For the first, multiply top and bottom by the conjugate $\sqrt{x+h} + \sqrt{x}$: the numerator becomes $(x+h) - x = h$, and canceling h leaves $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$. So

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$$\frac{f(x+h)-f(x)}{h} = \frac{1}{\sqrt{x+h}+\sqrt{x}} - 2.$$

Check with $x = 1$, $h = 3$: $f(1) = -1$ and $f(4) = -6$, average rate $-\frac{5}{3}$; the formula gives $\frac{1}{3} - 2 = -\frac{5}{3}$.

SCORING

1 point for multiplying by the conjugate to get $\frac{1}{\sqrt{x+h}+\sqrt{x}}$; 1 point for the full quotient with the -2.

3. A function f has $f(3) = -7$, and its rate of change at $x = 3$ is -2. Write the tangent line to the graph of f at $x = 3$ in the form $y = mx + b$, and use it to estimate $f(3.1)$. (2 points)

ANSWER

$y = -2x - 1$; $f(3.1) \approx -7.2$

SOLUTION

The tangent line passes through $(3, -7)$ and its slope is the rate, -2:

$$y = -7 + (-2)(x - 3) = -2x - 1.$$

Near 3 the line estimates f well. At $x = 3.1$: $y = -7 + (-2)(0.1) = -7.2$, so $f(3.1) \approx -7.2$.

4. Let $f(x) = \dfrac{x}{x^2+4}$. (3 points)

(a) Find and simplify the difference quotient of f, and say which values of x and $x + h$ it needs.

(b) Use it to find the average rate of change of f on $[0, 2]$.

ANSWER

(a) $\dfrac{4 - x^2 - xh}{(x^2+4)((x+h)^2+4)}$; every x and $x + h$ are allowed (only $h \neq 0$), since $x^2 + 4$ is never 0. (b) $\frac{1}{8}$

SOLUTION

(a) $f(x+h) = \dfrac{x+h}{(x+h)^2+4}$. Over the common denominator $(x^2+4)\left((x+h)^2+4\right)$ the numerator of $f(x+h) - f(x)$ is

$$\begin{aligned}&(x+h)(x^2+4) - x\left((x+h)^2+4\right)\\ &\quad= (x^3 + x^2h + 4x + 4h) - (x^3 + 2x^2h + xh^2 + 4x)\\ &\quad= 4h - x^2h - xh^2 = h(4 - x^2 - xh).\end{aligned}$$

Dividing by $h \neq 0$ cancels the h:

$$\frac{f(x+h)-f(x)}{h} = \frac{4 - x^2 - xh}{(x^2+4)((x+h)^2+4)}.$$

Each denominator factor is at least 4, so both ends are always in the domain: every x and $x + h$ work, with $h \neq 0$.

(b) The interval starts at $x = 0$ and has width $h = 2$: $\dfrac{4 - 0 - 0}{(4)(8)} = \frac{1}{8}$. Check: $f(0) = 0$ and $f(2) = \frac{2}{8} = \frac{1}{4}$, so $\dfrac{\frac{1}{4} - 0}{2} = \frac{1}{8}$.

SCORING

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1 point for the common denominator and the numerator $h(4 - x^2 - xh)$; 1 point for the simplified quotient with no restriction on x and $x + h$; 1 point for $\frac{1}{8}$ in (b).

5. Let $f(x) = \sqrt{4 - x}$. (3 points)

(a) Explain why there are no right estimates of the rate of f at $x = 4$.

(b) Find the left estimates $\frac{f(4 + h) - f(4)}{h}$ with $h = -0.01$ and $h = -0.0001$.

(c) Does f have a rate of change at $x = 4$? Explain.

ANSWER

(a) For $h > 0$, $4 + h$ is not in the domain. (b) −10 and −100. (c) No: only the left side exists, and the left estimates grow without bound and approach no number.

SOLUTION

(a) $f(x)$ needs $4 - x \geq 0$, so the domain is $x \leq 4$. A right estimate needs $f(4 + h)$ with $h > 0$, and $4 + h > 4$ is outside the domain.

(b) $f(4) = 0$. With $h = -0.01$: $f(3.99) = \sqrt{0.01} = 0.1$, so $\frac{0.1 - 0}{-0.01} = -10$. With $h = -0.0001$: $f(3.9999) = \sqrt{0.0001} = 0.01$, so $\frac{0.01 - 0}{-0.0001} = -100$.

(c) No. A rate at 4 needs the function defined on both sides, with the right and the left estimates approaching one number. Here there are no right estimates at all, and the left ones, $\frac{\sqrt{-h}}{h} = -\frac{1}{\sqrt{-h}}$, grow without bound as h shrinks, so they approach no number either.

SCORING

1 point: $4 + h$ lies outside the domain $x \leq 4$ for $h > 0$. 1 point: left estimates −10 and −100. 1 point: no rate, because one side is missing (and the left estimates approach no number).

6. Amara measures the height of a sunflower every 3 days.

day	0	3	6	9	12
height (cm)	150	177	198	213	225

Find the average growth rate over each step, in centimeters per day. Then describe the height with "increasing" or "decreasing" and "concave up" or "concave down". (2 points)

ANSWER

9, 7, 5, 4 cm per day; increasing and concave down (the sunflower grows by less each step).

SOLUTION

The first differences are 27, 21, 15 and 12 cm. Dividing by the step of 3 days gives 9, 7, 5 and 4 cm per day.

The rates are positive, so the height is increasing. The rates decrease, so the table is concave down: the sunflower is still growing, but more slowly.

SCORING

1 point for the four rates in centimeters per day; 1 point for "increasing and concave down".

7. The table gives a quadratic function $f(x) = px^2 + qx + c$ at equally spaced inputs.

x	0	2	4	6
$f(x)$	−1	1	35	101

Use differences to find p, and say whether f is concave up or concave down. (2 points)

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ANSWER

$p = 4$; f is concave up.

SOLUTION

The first differences are 2, 34 and 66, and the second differences are both 32.

With inputs spaced by $d = 2$, the first differences of a quadratic change by $2pd^2$ each step. So $2p(4) = 32$ and $p = \dfrac{32}{8} = 4$.

The second differences are positive, so f is concave up.

SCORING

1 point for the second difference 32; 1 point for $p = 4$ with the concavity.

8. Mei times an elevator as it starts up from the lobby of an office tower. Her reading at $t = 6$ was lost. (3 points)

t (seconds)	0	2	4	6	8
height (meters)	0.0	3.0	8.4	lost	22.2

(a) Find the left estimate of the elevator's speed at $t = 4$ with width 2.

(b) Before the reading was lost, Mei found that the right estimate at $t = 4$ with width 2 was 3.3 meters per second. Find the lost height.

(c) Find the centered average at $t = 4$ with $h = 2$. Of the left estimate, the right estimate and the centered average, which is the best estimate of the speed at $t = 4$? Explain.

ANSWER

(a) 2.7 meters per second. (b) 15.0 meters. (c) 3.0 meters per second; the centered average is the best estimate.

SOLUTION

(a) Left: $\dfrac{8.4 - 3.0}{2} = 2.7$ meters per second.

(b) The right estimate is $\dfrac{y - 8.4}{2} = 3.3$, so $y - 8.4 = 6.6$ and the lost height is $y = 15.0$ meters.

(c) $\dfrac{15.0 - 3.0}{4} = 3.0$ meters per second, the mean of 2.7 and 3.3. The elevator is speeding up, so the right estimate is too high and the left one too low by nearly equal amounts. The centered average is their mean and cancels most of both errors, so the best estimate is about 3.0 meters per second.

SCORING

1 point: the left estimate 2.7. 1 point: the lost height 15.0, from 8.4 + 2(3.3). 1 point: the centered average 3.0, chosen as best because the one-sided errors are nearly opposite and the mean cancels most of them.

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