Precalculus Unit 1 quiz 3: Slant asymptotes, Polynomial and rational inequalities with sign charts (answer key)

8 questions, 20 points

ANSWERS AT A GLANCE

1. (a) $y = x + 4$; $x = -2$ (b) $y = 3x - 1$; $x = 2$
2. $c > -4$
3. $(-6, 0) \cup (0, 3)$
4. $(-\infty, 2]$
5. (a) $y = 2x - \frac{5}{2}$, since $f(x) = 2x - \frac{5}{2} + \frac{7}{2(2x+1)}$. (b) The + 1 in the denominator: in the division it is multiplied by the quotient's $2x$ and changes the x term, which moves the intercept from $-\frac{3}{2}$ to $-\frac{5}{2}$. The + 1 in the numerator only changes the remainder.
6. (a) $b = -4$ (the slant asymptote is $y = 2x$). (b) Above the line on the right, below it on the left.
7. $\left(-1, -\frac{1}{2}\right] \cup (1, \infty)$
8. (a) For example $\frac{x-3}{x+2} \geq 0$. (b) $\frac{x-3}{x+2} \leq 0$; −2 makes the denominator zero, so the expression has no value there.

WORKED SOLUTIONS

1. Find the slant asymptote and the vertical asymptote of each function. (2 points)

(a) $\frac{x^2 + 6x + 1}{x + 2}$ (b) $\frac{3x^2 - 7x + 5}{x - 2}$

ANSWER

(a) $y = x + 4$; $x = -2$ (b) $y = 3x - 1$; $x = 2$

SOLUTION

(a) $x(x+2) = x^2 + 2x$ leaves $4x + 1$, and $4(x+2) = 4x + 8$ leaves −7. So the function is $x + 4 - \frac{7}{x+2}$: slant asymptote $y = x + 4$. The numerator at $x = -2$ is −7 ≠ 0, so $x = -2$ is the vertical asymptote.

(b) $3x(x-2) = 3x^2 - 6x$ leaves $-x + 5$, and $-1(x-2) = -x + 2$ leaves 3. So the function is $3x - 1 + \frac{3}{x-2}$: slant asymptote $y = 3x - 1$. The numerator at $x = 2$ is 3 ≠ 0, so $x = 2$ is the vertical asymptote.

SCORING

1 point for each function with both asymptotes correct.

2. Let $k(x) = \frac{x^2 + c}{x - 2}$, where c is a constant. For which values of c is the graph of k above its slant asymptote far to the right? (2 points)

ANSWER

$c > -4$

SOLUTION

Divide: $x(x-2) = x^2 - 2x$ leaves $2x + c$, and $2(x-2) = 2x - 4$ leaves $c + 4$. So

$$k(x) = x + 2 + \frac{c+4}{x-2}.$$

Far to the right $x - 2 > 0$, so the graph is above the line $y = x + 2$ exactly when $c + 4 > 0$, that is $c > -4$. (When $c = -4$ the remainder is 0 and the graph is the line itself with a hole at (2, 4); when $c < -4$ it is below the line on the right.)

SCORING

1 point for the remainder $c + 4$ (with the slant asymptote $y = x + 2$); 1 point for $c > -4$.

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3. Solve $18x^2 - 3x^3 > x^4$. Write the answer in interval notation. (2 points)

ANSWER

$(-6, 0) \cup (0, 3)$

SOLUTION

Subtract x^4 and factor out x^2 (dividing by x^2 would lose track of $x = 0$):

$$x^2(-x^2 - 3x + 18) = x^2(x + 6)(3 - x) > 0.$$

The critical numbers are −6, 0 and 3. Far to the right the factor $3 - x$ is negative and the others are positive, so the product is $-$ on $(3, \infty)$. It flips at 3 to $+$ on (0, 3), stays $+$ at 0 because x^2 has an even exponent, so $+$ on (−6, 0), and flips at −6 to $-$ on $(-\infty, -6)$.

The product is positive on (−6, 0) and (0, 3). The inequality is strict, so every zero is out, including 0, where the product is 0:

$$(-6, 0) \cup (0, 3).$$

4. Solve $x^3 \leq 3x + 2$. Start by finding a whole-number zero of $x^3 - 3x - 2$. (2 points)

ANSWER

$(-\infty, 2]$

SOLUTION

Subtract $3x + 2$: $x^3 - 3x - 2 \leq 0$. At $x = 2$, 8 − 6 − 2 = 0, so $x - 2$ is a factor. Synthetic division by 2 on 1, 0, −3, −2 gives 1, 2, 1 with remainder 0:

$$x^3 - 3x - 2 = (x - 2)(x^2 + 2x + 1) = (x - 2)(x + 1)^2.$$

The sign is $+$ on $(2, \infty)$, flips at 2 to $-$ on (−1, 2), and does not flip at −1 (even exponent), so it is $-$ on $(-\infty, -1)$ too. With the zeros −1 and 2 included, the pieces join into one interval: $(-\infty, 2]$.

5. Samir wants the slant asymptote of $f(x) = \dfrac{4x^2 - 3x + 1}{2x + 1}$. He says: "Far out, the constant terms do not matter, so $f(x) \approx \dfrac{4x^2 - 3x}{2x} = 2x - \dfrac{3}{2}$, and the slant asymptote is $y = 2x - \dfrac{3}{2}$." (3 points)

(a) Find the slant asymptote by division.

(b) Which of the two constants Samir dropped changes the answer, and why?

ANSWER

(a) $y = 2x - \dfrac{5}{2}$, since $f(x) = 2x - \dfrac{5}{2} + \dfrac{7}{2(2x + 1)}$. (b) The + 1 in the denominator: in the division it is multiplied by the quotient's $2x$ and changes the x term, which moves the intercept from $-\dfrac{3}{2}$ to $-\dfrac{5}{2}$. The + 1 in the numerator only changes the remainder.

SOLUTION

(a) $4x^2 \div 2x = 2x$, and $2x(2x + 1) = 4x^2 + 2x$ leaves $-5x + 1$; then $-5x \div 2x = -\dfrac{5}{2}$, and $-\dfrac{5}{2}(2x + 1) = -5x - \dfrac{5}{2}$ leaves $\dfrac{7}{2}$. So

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$$f(x) = 2x - \frac{5}{2} + \frac{7}{2(2x+1)},$$

and the slant asymptote is $y = 2x - \frac{5}{2}$.

(b) The numerator's + 1 only reaches the remainder, which shrinks far out, so dropping it is harmless. The denominator's + 1 is not harmless: in the first step it is multiplied by $2x$, and the $2x$ it makes is subtracted from $-3x$, giving $-5x$ instead of $-3x$. That changes the second term of the quotient. Samir's line is off by 1: the gap $f(x) - \left(2x - \frac{3}{2}\right)$ settles toward −1, not 0.

SCORING

1 point for the division with quotient $2x - \frac{5}{2}$; 1 point for the slant asymptote $y = 2x - \frac{5}{2}$; 1 point for naming the denominator's constant and explaining that it changes the quotient (the numerator's constant only changes the remainder).

6. Let $f(x) = \dfrac{6x^2 + bx + 1}{3x - 2}$. (3 points)

(a) Find b so that the slant asymptote of f passes through the origin.

(b) For that b, say which side of the slant asymptote the graph is on far to the right and far to the left.

ANSWER

(a) $b = -4$ (the slant asymptote is $y = 2x$). (b) Above the line on the right, below it on the left.

SOLUTION

(a) Divide: $6x^2 \div 3x = 2x$, and $2x(3x - 2) = 6x^2 - 4x$ leaves $(b + 4)x + 1$; the next quotient term is $(b + 4)x \div 3x = \dfrac{b+4}{3}$. So the slant asymptote is $y = 2x + \dfrac{b+4}{3}$. It passes through the origin when $\dfrac{b+4}{3} = 0$, so $b = -4$.

(b) With $b = -4$, $6x^2 - 4x + 1 = 2x(3x - 2) + 1$, so $f(x) = 2x + \dfrac{1}{3x-2}$. Far to the right $3x - 2 > 0$, so the gap $\dfrac{1}{3x-2}$ is positive and the graph is above $y = 2x$. Far to the left $3x - 2 < 0$, so the gap is negative and the graph is below the line.

SCORING

1 point for the slant asymptote $y = 2x + \dfrac{b+4}{3}$ (or the division that leads to it); 1 point for $b = -4$; 1 point for both sides correct with the sign of $\dfrac{1}{3x-2}$ as the reason.

7. Solve $\dfrac{x+5}{x-1} \geq \dfrac{x-1}{x+1}$. (3 points)

ANSWER

$\left(-1, -\dfrac{1}{2}\right] \cup (1, \infty)$

SOLUTION

Subtract the right side and use the common denominator $(x - 1)(x + 1)$:

$$\frac{(x+5)(x+1) - (x-1)^2}{(x-1)(x+1)} = \frac{x^2 + 6x + 5 - x^2 + 2x - 1}{(x-1)(x+1)} = \frac{4(2x+1)}{(x-1)(x+1)} \geq 0.$$

The numerator is zero at $-\dfrac{1}{2}$; the denominator is zero at −1 and 1. All exponents are 1 and the leading

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coefficients are positive, so from the right the signs are $+$ on $(1, \infty)$, $-$ on $\left(-\frac{1}{2}, 1\right)$, $+$ on $\left(-1, -\frac{1}{2}\right)$ and $-$ on $(-\infty, -1)$.

Include the zero $-\frac{1}{2}$; leave out -1 and 1, where the expression is undefined:

$$\left(-1, -\frac{1}{2}\right] \cup (1, \infty).$$

Check: $x = 0$ gives $-5 \geq -1$, false, and 0 is not in the answer.

SCORING

1 point: one fraction compared with 0, numerator $8x + 4$ (or $4(2x + 1)$) over $(x - 1)(x + 1)$. 1 point: critical numbers -1, $-\frac{1}{2}$, 1 with the correct sign on each interval. 1 point: correct answer, with $-\frac{1}{2}$ included and -1, 1 excluded.

8. (3 points)

(a) Write a rational inequality whose solution set is $(-\infty, -2) \cup [3, \infty)$.

(b) Change only the inequality sign in your answer to (a) so that the solution set becomes $(-2, 3]$. Explain why -2 is left out of both sets.

ANSWER

(a) For example $\frac{x - 3}{x + 2} \geq 0$. (b) $\frac{x - 3}{x + 2} \leq 0$; -2 makes the denominator zero, so the expression has no value there.

SOLUTION

(a) The endpoint 3 is included, so it should be a zero of the numerator with $\geq$; the endpoint -2 is excluded even though the sign is not strict, so it should be a zero of the denominator. Try $\frac{x - 3}{x + 2} \geq 0$. From the right the signs are $+$ on $(3, \infty)$, $-$ on $(-2, 3)$, $+$ on $(-\infty, -2)$. Positive pieces plus the zero 3: $(-\infty, -2) \cup [3, \infty)$.

(b) $\frac{x - 3}{x + 2} \leq 0$ keeps the negative stretch $(-2, 3)$ and the zero 3: $(-2, 3]$. In both sets -2 is out because $x + 2 = 0$ there, and a fraction with denominator 0 is undefined, so it cannot satisfy any inequality.

SCORING

1 point: an inequality in (a) with exactly the set $(-\infty, -2) \cup [3, \infty)$ (any correct one, for example with $x + 2$ in the denominator and $x - 3$ in the numerator). 1 point: the reversed sign in (b) giving exactly $(-2, 3]$. 1 point: the reason: -2 is a zero of the denominator, so the expression is undefined there.

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