Precalculus Unit 1 test: Functions in depth: rate of change, end behavior and asymptotes (answer key)

12 questions, 51 points

ANSWERS AT A GLANCE

1. (a) $p(x) = x(x-2)(x+2)(x^2+4)$; zeros 0, 2, −2, $2i$, $-2i$ (b) as $x \to -\infty$, $p(x) \to -\infty$, and as $x \to \infty$, $p(x) \to \infty$; range all real numbers (c) odd

2. (a) as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$ (b) one zero with $-2 < x < 0$, one with $0 < x < 2$ and one with $x > 2$ (c) a valley with $-2 < x < 2$ and a peak with $x > 0$; at most 4 (d) all real numbers

3. (a) $f(x) = \dfrac{(x+1)(x+2)(x-3)}{(x+2)(x-1)}$; every real number except −2 and 1. (b) Hole at $\left(-2, -\dfrac{5}{3}\right)$; vertical asymptote $x = 1$: $f(x) \to \infty$ as $x \to 1^-$, $f(x) \to -\infty$ as $x \to 1^+$. (c) $y = x - 1$; below it far right, above it far left. (d) Zeros $x = -1$ and $x = 3$, y-intercept (0, 3); − on $(-\infty, -2)$ and (−2, −1), + on (−1, 1), − on (1, 3), + on $(3, \infty)$.

4. (a) $x = 0$: up on the left, down on the right; $x = 4$: down on the left, up on the right; $y = 1$. Touches the axis at (2, 0); no y-intercept. + on $(-\infty, 0)$, − on (0, 2) and (2, 4), + on $(4, \infty)$. (b) Leading coefficients 1 and 1; setting $f(x) = 1$ and clearing the denominator leaves $(k+2)x = 2k$. (c) $k = -4$, $k = -2$, $k = 0$.

5. (a) $3x^2 + 3xh + h^2 + 4x + 2h - 6$ (b) With $x = -w$ and $h = 2w$ the quotient is $w^2 - 6$. (c) $[-4, 4]$

6. (a) $\dfrac{2n+h}{100} + 2$ dollars per poster, the average cost of each extra poster when the order grows from n to $n + h$ posters. (b) 5 dollars per poster (c) 7 dollars per poster, 2 dollars more

7. (a) $a = 1$ (b) $b = 7$; they meet at (−3, −5). (c) Above the line.

8. (a) $p(x) = (x-2)^2(x-1)(x+1)(x+2)$ (b) −2, −1, 1, each of multiplicity 1, crosses; 2, multiplicity 2, touches and turns back; as $x \to -\infty$, $p(x) \to -\infty$, and as $x \to \infty$, $p(x) \to \infty$; y-intercept (0, −8) (c) negative for $x < -2$, positive on $-2 < x < -1$, negative on $-1 < x < 1$, positive on $1 < x < 2$, positive for $x > 2$; $p\left(-\dfrac{3}{2}\right) = \dfrac{245}{32}$, $p\left(\dfrac{3}{2}\right) = \dfrac{35}{32}$ (d) a peak in $-2 < x < -1$, a valley in $-1 < x < 1$, a peak in $1 < x < 2$, a valley at (2, 0); four turns is $n - 1$. Features to grade on the sketch: slanted crossings at −2, −1 and 1; a touch from above at (2, 0); left end down, right end up; through (0, −8); the four turns in the right intervals and no others.

9. (a) $1 + 6h + 4h^2 + h^3$ and $1 + 4h^2$. (b) The rate is 1; the right estimates lie above it and the left estimates below it. (c) 1.060401, off by 0.060401; 1.0004, off by 0.0004. (d) $y = x - 3$.

10. (a) $x \geq 60$, the interval $[60, \infty)$. (b) $R(x) - 15 = \dfrac{-225}{15 + x}$, which is negative for every $x > 0$.

11. (a) at $x = 0$, $x = 2$ and $x = 7$ (the entrance, 200 m in and 700 m in); it passes below only at $x = 7$ (b) above for $0 < x < 2$ and $2 < x < 7$, below for $7 < x \leq 8$ (c) degree 4 allows three turns, and a peak in $0 < x < 2$ and the valley at $x = 2$ use two of them, so only one peak fits in $2 < x < 7$; $E(5) = 22.5$, $E(5.5) \approx 25.3$, $E(6) = 24$; the peak lies between 500 m and 600 m in, estimated 25.3 m above the entrance (d) $E(8) = -72$: at the far end the floor is 72 m below the entrance.

12. (a) $b = 9$, $c = 85$. (b) Concave up on (0, 3), concave down on (3, 6); point of inflection (3, 309): 3,090 visitors had come in by the busiest moment. (c) Yes: the rate at $t = 6$ is 85 tens, or 850 visitors, per hour.

WORKED SOLUTIONS

1. Let $p(x) = x^5 - 16x$. (4 points)

(a) Factor $p(x)$ completely over the integers, and find all five of its zeros, real and non-real.

(b) Describe the end behavior of p in arrow notation, and give its range.

(c) Is p even, odd or neither? Show the work that decides it.

ANSWER

(a) $p(x) = x(x-2)(x+2)(x^2+4)$; zeros 0, 2, −2, $2i$, $-2i$ (b) as $x \to -\infty$, $p(x) \to -\infty$, and as $x \to \infty$, $p(x) \to \infty$; range all real numbers (c) odd

SOLUTION

(a) Take out the common factor, then use the difference of squares twice where it applies:

$$x^5 - 16x = x(x^4 - 16) = x(x^2 - 4)(x^2 + 4) = x(x-2)(x+2)(x^2+4).$$

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The real zeros are 0, 2 and −2. The factor $x^2 + 4$ is 0 where $x^2 = -4$, at $x = 2i$ and $x = -2i$. That is five zeros for degree 5, three of them real.

(b) The leading term x^5 has odd degree and a positive coefficient: as $x \to -\infty$, $p(x) \to -\infty$, and as $x \to \infty$, $p(x) \to \infty$. A polynomial of odd degree has range all real numbers.

(c) $p(-x) = (-x)^5 - 16(-x) = -x^5 + 16x = -(x^5 - 16x) = -p(x)$, so p is odd: its graph has half-turn symmetry about the origin.

SCORING

1 point: complete factorization $x(x-2)(x+2)(x^2+4)$. 1 point: all five zeros, including $\pm 2i$. 1 point: correct end behavior and range all real numbers. 1 point: $p(-x) = -p(x)$ shown, so p is odd.

2. A polynomial p has degree 5 and a negative leading coefficient. Its formula is not given, but $p(-2) = 3$, $p(0) = -1$ and $p(2) = 4$. (4 points)

(a) Describe the end behavior of p in arrow notation.

(b) Explain why p has at least three real zeros, and say in which interval each one lies.

(c) Explain why the graph of p has at least two turning points. What is the greatest number of turning points it can have?

(d) Give the range of p.

ANSWER

(a) as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$ (b) one zero with $-2 < x < 0$, one with $0 < x < 2$ and one with $x > 2$ (c) a valley with $-2 < x < 2$ and a peak with $x > 0$; at most 4 (d) all real numbers

SOLUTION

(a) Odd degree and a negative leading coefficient: the left end rises and the right end falls. As $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$.

(b) A polynomial graph has no breaks, so it crosses the x-axis wherever its outputs change sign. From $p(-2) = 3$ to $p(0) = -1$ the sign changes: a zero with $-2 < x < 0$. From $p(0) = -1$ to $p(2) = 4$: a zero with $0 < x < 2$. By (a), far to the right the outputs are negative, so from $p(2) = 4$ to such a point the sign changes again: a zero with $x > 2$. The three intervals do not overlap, so these are three different zeros.

(c) On $-2 \le x \le 2$ the graph has a lowest point. Its output is at most $p(0) = -1$, lower than the values 3 and 4 at the two ends of the interval, so it is not at an end: the graph falls into it and rises out of it, a valley with $-2 < x < 2$. Next pick a point $R > 2$ far enough right that $p(R) < -1$, as (a) allows. On $0 \le x \le R$ the graph has a highest point, with output at least $p(2) = 4$, higher than $p(0) = -1$ and $p(R)$, so it is not at an end either: a peak with $0 < x < R$. The valley has output at most −1 and the peak at least 4, so they are two different turning points. A polynomial of degree 5 has at most 5 − 1 = 4 turning points.

(d) The degree is odd, so the range is all real numbers.

SCORING

1 point: correct end behavior in arrow notation, and range all real numbers. 1 point: three sign changes named, on $-2 < x < 0$, $0 < x < 2$ and $x > 2$, with no breaks as the reason (the last one needs the falling right end). 1 point: a valley between −2 and 2 and a peak to the right of 0, each with a reason, and the two shown to be different. 1 point: at most 4 turning points, from degree 5.

3. Let $f(x) = \dfrac{x^3 - 7x - 6}{x^2 + x - 2}$. The numerator is 0 at $x = -1$. (4 points)

(a) Factor the numerator and the denominator completely, and give the domain of f.

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(b) Find every hole and vertical asymptote, and say what the graph does on each side of each vertical asymptote.

(c) Find the line the graph follows far out, and say on which side of it the graph lies far to the right and far to the left.

(d) Find the intercepts and the sign of f on each interval, and sketch the graph.

ANSWER

(a) $f(x) = \dfrac{(x+1)(x+2)(x-3)}{(x+2)(x-1)}$; every real number except −2 and 1. (b) Hole at $\left(-2, -\dfrac{5}{3}\right)$; vertical asymptote $x = 1$: $f(x) \to \infty$ as $x \to 1^-$, $f(x) \to -\infty$ as $x \to 1^+$. (c) $y = x - 1$; below it far right, above it far left. (d) Zeros $x = -1$ and $x = 3$, y-intercept (0, 3); $-$ on $(-\infty, -2)$ and (−2, −1), $+$ on (−1, 1), $-$ on (1, 3), $+$ on $(3, \infty)$.

SOLUTION

(a) Since the numerator is 0 at −1, the factor theorem says $x + 1$ is a factor. Synthetic division of 1, 0, −7, −6 by −1 gives 1, −1, −6 with remainder 0, so $x^3 - 7x - 6 = (x+1)(x^2 - x - 6) = (x+1)(x+2)(x-3)$. The denominator is $(x+2)(x-1)$. So

$$f(x) = \frac{(x+1)(x+2)(x-3)}{(x+2)(x-1)} = \frac{(x+1)(x-3)}{x-1} \quad (x \neq -2).$$

The domain is every real number except −2 and 1.

(b) The factor $x + 2$ cancels completely, so $x = -2$ is a hole, at height $\dfrac{(-1)(-5)}{-3} = -\dfrac{5}{3}$. The lowest-terms denominator is zero at 1, an odd power, so $x = 1$ is a vertical asymptote with opposite directions on its sides. $C = (2)(-2) = -4 < 0$: as $x \to 1^-$, $x - 1$ is a small negative number and $f(x) \to \infty$; as $x \to 1^+$, $f(x) \to -\infty$.

(c) After canceling, the numerator $x^2 - 2x - 3$ has degree one more than $x - 1$, so there is no horizontal asymptote. Divide: $x^2 - 2x - 3 = (x-1)(x-1) - 4$, so $f(x) = x - 1 - \dfrac{4}{x-1}$ for $x \neq -2$, and the graph follows the slant line $y = x - 1$. Far right $\dfrac{4}{x-1}$ is a small positive number being subtracted, so the graph is just below the line; far left it is a small negative number being subtracted, so the graph is just above. The gap $-\dfrac{4}{x-1}$ is never 0, so the graph never meets the line.

(d) The zeros are −1 and 3 (not −2, which is a hole), both crossings; $f(0) = \dfrac{(1)(-3)}{-1} = 3$. The breaks −1, 1, 3 are all odd powers and the far right is $+$, so the signs alternate: $+$ on $(3, \infty)$, $-$ on (1, 3), $+$ on (−1, 1), $-$ on $(-\infty, -1)$, which the hole splits into $(-\infty, -2)$ and (−2, −1) without changing the sign.

Grade the sketch on: dashed $x = 1$ and $y = x - 1$; a left branch coming up from the lower left just above the slant line, through (−3, −3), with an open circle at $\left(-2, -\dfrac{5}{3}\right)$, crossing at (−1, 0), through (0, 3) and up to ∞ along $x = 1$; a right branch up from $-\infty$ along $x = 1$, through (2, −3), crossing at (3, 0), through $\left(4, \dfrac{5}{3}\right)$ and rising to the upper right just below the slant line.

SCORING

1 point: complete factoring of the numerator (factor theorem or division) and the domain. 1 point: the hole with its height, and the vertical asymptote $x = 1$ with both side behaviors. 1 point: the slant line $y = x - 1$ by division, with the side far right and far left. 1 point: both intercepts, the signs on every interval, and a sketch showing the hole as an open circle and matching every feature.

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4. Let $f(x) = \dfrac{(x-2)(x+k)}{x(x-4)}$, where k is a constant. (4 points)

(a) For $k = -2$, find the asymptotes, what the graph does on each side of each vertical asymptote, the intercepts and the signs, and sketch the graph.

(b) Show that for every k the horizontal asymptote is $y = 1$, and that the graph can meet it only where $(k+2)x = 2k$.

(c) Find every value of k for which the graph never meets its horizontal asymptote. Explain each.

ANSWER

(a) $x = 0$: up on the left, down on the right; $x = 4$: down on the left, up on the right; $y = 1$. Touches the axis at (2, 0); no y-intercept. $+$ on $(-\infty, 0)$, $-$ on (0, 2) and (2, 4), $+$ on $(4, \infty)$. (b) Leading coefficients 1 and 1; setting $f(x) = 1$ and clearing the denominator leaves $(k+2)x = 2k$. (c) $k = -4$, $k = -2$, $k = 0$.

SOLUTION

(a) For $k = -2$, $f(x) = \dfrac{(x-2)^2}{x(x-4)}$; nothing cancels. Vertical asymptotes $x = 0$ and $x = 4$, odd powers. At 0, $C = \dfrac{4}{-4} = -1 < 0$: $f(x) \to \infty$ as $x \to 0^-$, $f(x) \to -\infty$ as $x \to 0^+$. At 4, $C = \dfrac{4}{4} = 1 > 0$: $f(x) \to -\infty$ as $x \to 4^-$, $f(x) \to \infty$ as $x \to 4^+$. Equal degrees, leading coefficients 1 and 1: $y = 1$. The zero is $x = 2$, an even power, so the graph touches the axis at (2, 0). There is no y-intercept, since 0 is not in the domain. Signs: $+$ on $(4, \infty)$, flip at 4: $-$ on (2, 4), stay at 2: $-$ on (0, 2), flip at 0: $+$ on $(-\infty, 0)$. The gap $f(x) - 1 = \dfrac{(x-2)^2 - x(x-4)}{x(x-4)} = \dfrac{4}{x(x-4)}$ is never 0 and is positive outside $[0, 4]$, so both outer branches stay above $y = 1$.

Grade the sketch on: dashed $x = 0$, $x = 4$, $y = 1$; a left branch just above $y = 1$ far left, rising to ∞ along $x = 0$; a middle branch below the axis, down to $-\infty$ along both $x = 0$ and $x = 4$, touching the axis at its top, (2, 0); a right branch down from ∞ along $x = 4$, flattening toward $y = 1$ from above.

(b) Expanded, $f(x) = \dfrac{x^2 + (k-2)x - 2k}{x^2 - 4x}$: equal degrees, leading coefficients 1 and 1, so $y = 1$ for every k (canceling a common factor, when there is one, leaves degree 1 over degree 1, still with ratio 1). The graph meets $y = 1$ where $f(x) = 1$; multiplying by $x(x-4)$ gives $x^2 + (k-2)x - 2k = x^2 - 4x$, so $(k+2)x = 2k$, and only solutions in the domain count.

(c) If $k = -2$ the equation reads $0 = -4$, which no x satisfies: no meeting. Otherwise its one solution is $x = \dfrac{2k}{k+2}$, which is a meeting point unless it is refused, that is unless it equals 0 or 4. It is 0 when $k = 0$, and 4 when $2k = 4k + 8$, $k = -4$. In both cases the factor of the refused input cancels (x when $k = 0$, $x - 4$ when $k = -4$), so the graph has a hole there, at height $\dfrac{1}{2}$, not 1. So the graph never meets $y = 1$ exactly when $k = -4$, $k = -2$ or $k = 0$.

SCORING

2 points for (a): 1 for both vertical asymptotes with side behaviors and $y = 1$; 1 for the touch at (2, 0), no y-intercept, the signs and a matching sketch. 1 point for (b) and (c) together up to $k = -2$: the equation $(k+2)x = 2k$ from $f(x) = 1$, and $k = -2$ because it then has no solution. 1 point: $k = 0$ and $k = -4$, with the reason that the only solution is a refused input.

5. Let $f(x) = x^3 + 2x^2 - 6x$. (4 points)

(a) Find and simplify the difference quotient of f.

(b) Use it to show that the average rate of change of f on any interval $[-w, w]$, with $w > 0$, is $w^2 - 6$.

(c) Find the interval $[-w, w]$ on which the average rate of change of f is 10, and check it from the values of f at the ends.

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ANSWER

(a) $3x^2 + 3xh + h^2 + 4x + 2h - 6$ (b) With $x = -w$ and $h = 2w$ the quotient is $w^2 - 6$. (c) $[-4, 4]$

SOLUTION

(a) $f(x+h) = x^3 + 3x^2h + 3xh^2 + h^3 + 2x^2 + 4xh + 2h^2 - 6x - 6h$. Subtracting $f(x) = x^3 + 2x^2 - 6x$ leaves $3x^2h + 3xh^2 + h^3 + 4xh + 2h^2 - 6h$. Every term has a factor h; canceling it, since $h \neq 0$:

$$\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2 + 4x + 2h - 6.$$

(b) The interval $[-w, w]$ starts at $x = -w$ and has width $h = 2w$:

$$3w^2 + 3(-w)(2w) + (2w)^2 + 4(-w) + 2(2w) - 6 = 3w^2 - 6w^2 + 4w^2 - 4w + 4w - 6 = w^2 - 6.$$

The x^2 term of f adds nothing on a symmetric interval, because $4x + 2h = 0$ there.

(c) $w^2 - 6 = 10$ gives $w^2 = 16$, and $w > 0$, so $w = 4$: the interval $[-4, 4]$.

Check: $f(4) = 64 + 32 - 24 = 72$ and $f(-4) = -64 + 32 + 24 = -8$, so $\frac{72 - (-8)}{8} = 10$.

SCORING

1 point for $f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3 + 4xh + 2h^2 - 6h$; 1 point for $3x^2 + 3xh + h^2 + 4x + 2h - 6$; 1 point for putting $x = -w$, $h = 2w$ and simplifying to $w^2 - 6$; 1 point for $[-4, 4]$ with the check.

6. Omar runs a print shop. Printing n posters of one design costs $C(n) = \frac{n^2}{100} + 2n + 200$ dollars. (4 points)

(a) Find and simplify $\frac{C(n+h) - C(n)}{h}$, and say what it measures, with units.

(b) An order grows from 100 to 200 posters. Use (a) to find the average cost of each extra poster.

(c) The order then grows by another 100 posters, to 300. Find the average cost of each of these extra posters, and how much more it is than in (b).

ANSWER

(a) $\frac{2n + h}{100} + 2$ dollars per poster, the average cost of each extra poster when the order grows from n to $n + h$ posters. (b) 5 dollars per poster (c) 7 dollars per poster, 2 dollars more

SOLUTION

(a) $C(n+h) - C(n) = \frac{(n+h)^2 - n^2}{100} + 2h = \frac{2nh + h^2}{100} + 2h$; the fixed 200 dollars cancel. Dividing by $h \neq 0$:

$$\frac{C(n+h) - C(n)}{h} = \frac{2n + h}{100} + 2$$

dollars per poster: the change in cost divided by the number of extra posters.

(b) $n = 100$ and $h = 100$: $\frac{300}{100} + 2 = 5$ dollars per poster. Check: $C(200) - C(100) = 1{,}000 - 500 = 500$ dollars, and 500 ÷ 100 = 5.

(c) $n = 200$ and $h = 100$: $\frac{500}{100} + 2 = 7$ dollars per poster, which is 2 dollars more than in (b). Each extra poster costs more on average as the order grows.

SCORING

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1 point for $C(n+h)-C(n)=\dfrac{2nh+h^2}{100}+2h$; 1 point for the simplified quotient with units of dollars per poster; 1 point for 5 dollars in (b); 1 point for 7 dollars and 2 dollars more in (c).

7. Let $f(x)=\dfrac{2x^3+ax^2+9x+b}{x^2+4}$, where a and b are constants. (4 points)

(a) Find a so that the slant asymptote of f is $y=2x+1$.

(b) With that a, find b so that the graph meets its slant asymptote at $x=-3$, and give the point where they meet.

(c) Which side of the slant asymptote is the graph on far to the right?

ANSWER

(a) $a=1$ (b) $b=7$; they meet at $(-3,-5)$. (c) Above the line.

SOLUTION

(a) Divide: $2x(x^2+4)=2x^3+8x$ leaves ax^2+x+b; then $a(x^2+4)=ax^2+4a$ leaves $x+(b-4a)$. So

$$f(x)=2x+a+\frac{x+b-4a}{x^2+4}.$$

The slant asymptote is $y=2x+a$, so $a=1$.

(b) With $a=1$ the remainder is $x+b-4$, and the denominator x^2+4 is never 0. The graph meets the line where the remainder is 0, so we need $-3+b-4=0$, which gives $b=7$. The remainder is then $x+3$, and the meeting point is on the line: $y=2(-3)+1=-5$, the point $(-3,-5)$. Check in the original: the top is $-54+9-27+7=-65$, the bottom is 13, and $-65\div13=-5$.

(c) The gap is $\dfrac{x+3}{x^2+4}$. The denominator is always positive, and far to the right $x+3>0$, so the graph is above the line.

SCORING

1 point for the division giving quotient $2x+a$ and $a=1$; 1 point for $b=7$ from setting the remainder to 0 at $x=-3$; 1 point for the point $(-3,-5)$; 1 point for "above" with the sign of $\dfrac{x+3}{x^2+4}$ as the reason.

8. Let $p(x)=x^5-2x^4-5x^3+10x^2+4x-8$. (5 points)

(a) Show that 2 is a zero of p, and factor $p(x)$ completely.

(b) Give each zero with its multiplicity and what the graph does there, the end behavior and the y-intercept.

(c) Give the sign of $p(x)$ on each interval between the zeros, and find $p\left(-\dfrac{3}{2}\right)$ and $p\left(\dfrac{3}{2}\right)$.

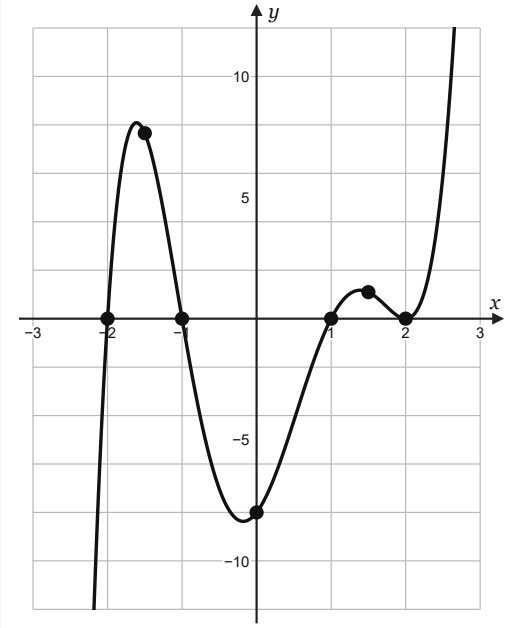
(d) Sketch the graph. List its peaks and valleys by interval, and explain why it can have no others.

ANSWER

(a) $p(x)=(x-2)^2(x-1)(x+1)(x+2)$ (b) -2, -1, 1, each of multiplicity 1, crosses; 2, multiplicity 2, touches and turns back; as $x\to-\infty$, $p(x)\to-\infty$, and as $x\to\infty$, $p(x)\to\infty$; y-intercept $(0,-8)$ (c) negative for $x<-2$, positive on $-2<x<-1$, negative on $-1<x<1$, positive on $1<x<2$, positive for $x>2$; $p\left(-\dfrac{3}{2}\right)=\dfrac{245}{32}$, $p\left(\dfrac{3}{2}\right)=\dfrac{35}{32}$ (d) a peak in $-2<x<-1$, a valley in $-1<x<1$, a peak in $1<x<2$, a valley at $(2,0)$; four turns is $n-1$.

Features to grade on the sketch: slanted crossings at -2, -1 and 1; a touch from above at $(2,0)$; left end down, right end up; through $(0,-8)$; the four turns in the right intervals and no others.

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SOLUTION

(a) $p(2) = 32 - 32 - 40 + 40 + 8 - 8 = 0$. Synthetic division of 1, −2, −5, 10, 4, −8 by 2 gives 1, 0, −5, 0, 4 with remainder 0, so $p(x) = (x-2)(x^4 - 5x^2 + 4)$. The quartic is a quadratic in x^2: $(x^2-1)(x^2-4)$. So

$$p(x) = (x-2)(x-1)(x+1)(x-2)(x+2) = (x-2)^2(x-1)(x+1)(x+2).$$

(Grouping also works: $x^4(x-2) - 5x^2(x-2) + 4(x-2)$.)

(b) 2 appears twice: a double zero, where the graph touches and turns back. −2, −1 and 1 are simple: it crosses. The leading term is x^5: the left end falls and the right end rises. $p(0) = (4)(-1)(1)(2) = -8$.

(c) For $x > 2$ every factor is positive. Moving left, the sign does not flip at the double zero 2 and flips at 1, −1 and −2: positive for $x > 2$, positive on $1 < x < 2$, negative on $-1 < x < 1$, positive on $-2 < x < -1$, negative for $x < -2$.

$p\left(-\frac{3}{2}\right) = \left(\frac{49}{4}\right)\left(-\frac{5}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{245}{32} \approx 7.7$ and $p\left(\frac{3}{2}\right) = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right) = \frac{35}{32} \approx 1.1$.

(d) From the bottom left the graph rises through −2 to a peak in $-2 < x < -1$ (at least 7.7 high), falls through −1 and (0, −8) to a valley in $-1 < x < 1$ (at least 8 deep), rises through 1 to a small peak in $1 < x < 2$, comes down to touch the axis at (2, 0), which is a valley, and rises off to the top right. That is four turns, and a polynomial of degree 5 has at most 5 − 1 = 4, so there are no others.

SCORING

1 point: $p(2) = 0$ shown and the division by $x - 2$. 1 point: complete factorization $(x-2)^2(x-1)(x+1)(x+2)$. 1 point: multiplicities with cross or touch, end behavior and y-intercept (0, −8). 1 point: the sign chart, with no flip at 2 (follow through from a wrong factorization). 1 point: a sketch with these features and the four turns, with the $n - 1$ reason that there are no others.

9. Let $f(x) = x^4 - 3x$. (5 points)

(a) Simplify the right estimate $\dfrac{f(1+h) - f(1)}{h}$ and the centered average $\dfrac{f(1+h) - f(1-h)}{2h}$.

(b) Find the rate of change of f at $x = 1$. For small widths, do the right estimates lie above or below it, and the left estimates?

(c) With $h = 0.01$, find the right estimate and the centered average, and how far each is from the rate.

(d) Write the tangent line to the graph of f at $x = 1$.

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ANSWER

(a) $1 + 6h + 4h^2 + h^3$ and $1 + 4h^2$. (b) The rate is 1; the right estimates lie above it and the left estimates below it. (c) 1.060401, off by 0.060401; 1.0004, off by 0.0004. (d) $y = x - 3$.

SOLUTION

(a) $f(1) = 1 - 3 = -2$. Expanding $(1 + h)^4 = 1 + 4h + 6h^2 + 4h^3 + h^4$, $f(1 + h) = 1 + 4h + 6h^2 + 4h^3 + h^4 - 3 - 3h = -2 + h + 6h^2 + 4h^3 + h^4$. So $f(1 + h) - f(1) = h + 6h^2 + 4h^3 + h^4$, and for $h \neq 0$ the right estimate is $1 + 6h + 4h^2 + h^3$.

Putting $-h$ for h, $f(1 - h) = -2 - h + 6h^2 - 4h^3 + h^4$. Subtracting, $f(1 + h) - f(1 - h) = 2h + 8h^3$, and dividing by $2h$ gives $1 + 4h^2$.

(b) As h shrinks, $1 + 6h + 4h^2 + h^3$ approaches 1 from both sides, so the rate at 1 is 1. The excess is $h(6 + 4h + h^2)$, and $6 + 4h + h^2$ is positive for small h, so the excess has the sign of h: the right estimates ($h > 0$) lie above 1 and the left estimates ($h < 0$) below.

(c) Right: 1 + 0.06 + 0.0004 + 0.000001 = 1.060401, off by 0.060401. Centered: 1 + 0.0004 = 1.0004, off by 0.0004. The one-sided estimate misses by about $6h$, the centered one by only $4h^2$.

(d) Through (1, −2) with slope 1: $y = -2 + 1(x - 1)$, that is $y = x - 3$.

SCORING

2 points: the right estimate $1 + 6h + 4h^2 + h^3$ and the centered average $1 + 4h^2$ (1 each). 1 point: the rate 1, with the right estimates above and the left estimates below. 1 point: 1.060401 off by 0.060401 and 1.0004 off by 0.0004. 1 point: the tangent line $y = x - 3$.

10. A 15-ohm resistor is wired in parallel with a second resistor of x ohms, where $x > 0$. The combined resistance is $R(x) = \dfrac{15x}{15 + x}$ ohms. (5 points)

(a) For which values of x is the combined resistance at least 12 ohms? Solve with a sign chart.

(b) Show that $R(x) < 15$ for every $x > 0$.

ANSWER

(a) $x \geq 60$, the interval $[\,60, \infty)$. (b) $R(x) - 15 = \dfrac{-225}{15 + x}$, which is negative for every $x > 0$.

SOLUTION

(a) Subtract 12 and use the denominator $15 + x$:

$$R(x) - 12 = \frac{15x - 12(15 + x)}{15 + x} = \frac{3x - 180}{15 + x} \geq 0.$$

The numerator is zero at $x = 60$; the denominator is zero at $x = -15$, which is not a resistance. For $x > 0$ the denominator is positive, so the sign is the sign of $3x - 180$: $-$ on $(0, 60)$ and $+$ on $(60, \infty)$. The inequality allows 0, so 60 is in: $x \geq 60$. Check: $R(60) = \dfrac{900}{75} = 12$.

(b) $R(x) - 15 = \dfrac{15x - 15(15 + x)}{15 + x} = \dfrac{-225}{15 + x}$. The numerator is a negative constant, never zero, and the denominator is positive for $x > 0$, so the fraction is negative on all of $(0, \infty)$. Hence $R(x) < 15$ for every $x > 0$.

SCORING

1 point: $R(x) - 12$ written as one fraction, $\dfrac{3x - 180}{15 + x}$. 1 point: critical number 60, and the sign on $(0, 60)$ and $(60, \infty)$ (using that the denominator is positive). 1 point: answer $x \geq 60$ with the endpoint included.

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1 point: (b) $R(x) - 15 = \dfrac{-225}{15 + x}$. 1 point: (b) the sign argument: negative numerator, positive denominator for $x > 0$.

11. Hiro surveys a cave passage that starts at the entrance and runs 800 m into a hillside. At x hundred meters along the passage, the floor is

$$E(x) = -\frac{1}{4}x(x-2)^2(x-7)$$

meters above the entrance, for $0 \le x \le 8$ (a negative value means below the entrance). (4 points)

(a) Where along the passage is the floor level with the entrance? At which of these places does the floor pass from above the entrance to below it?

(b) On which stretches of the passage is the floor above the entrance, and on which is it below?

(c) Explain why, between 200 m and 700 m in, the floor has exactly one peak. Find $E(5)$, $E(5.5)$ and $E(6)$, to one decimal place, and use them to name the 100 m stretch that contains this peak. Take the largest of the three values as your estimate of the peak's height.

(d) Find $E(8)$ and say what it means.

ANSWER

(a) at $x = 0$, $x = 2$ and $x = 7$ (the entrance, 200 m in and 700 m in); it passes below only at $x = 7$ (b) above for $0 < x < 2$ and $2 < x < 7$, below for $7 < x \le 8$ (c) degree 4 allows three turns, and a peak in $0 < x < 2$ and the valley at $x = 2$ use two of them, so only one peak fits in $2 < x < 7$; $E(5) = 22.5$, $E(5.5) \approx 25.3$, $E(6) = 24$; the peak lies between 500 m and 600 m in, estimated 25.3 m above the entrance (d) $E(8) = -72$: at the far end the floor is 72 m below the entrance.

SOLUTION

(a) $E(x) = 0$ when $x = 0$, $x = 2$ or $x = 7$. The zeros 0 and 7 are simple and 2 is double. At $x = 2$ (that is, 200 m in) the floor comes down to the entrance's level and goes back up without passing below: the graph touches. At $x = 7$ (700 m in) the zero is simple, so the floor crosses from above to below.

(b) For $0 < x < 2$: $x > 0$, $(x-2)^2 > 0$ and $x - 7 < 0$, so the product is negative and the factor $-\dfrac{1}{4}$ makes E positive. The sign does not change at the double zero 2, so E is positive on $2 < x < 7$ too, and it flips at 7: negative for $7 < x \le 8$. The floor is above the entrance all the way to 700 m (only level with it at 200 m) and below it from 700 m to the end.

(c) E has degree 4 (leading term $-\dfrac{1}{4}x^4$), so its graph has at most 3 turns. Between the neighboring zeros 0 and 2 the graph is above the axis, so it has a peak there; the touching zero (2, 0) is a valley; and between the neighboring zeros 2 and 7 it is above the axis, so it has at least one peak there. That is already three turns, so the peak in $2 < x < 7$ is the only turn there: the graph rises up to it and falls after it.
$E(5) = -\dfrac{1}{4}(5)(9)(-2) = 22.5$, $E(5.5) = -\dfrac{1}{4}(5.5)(12.25)(-1.5) \approx 25.3$ and $E(6) = -\dfrac{1}{4}(6)(16)(-1) = 24$.
If the peak were at or before $x = 5$, the graph would be falling from 5 to 5.5, but $E(5.5) > E(5)$; if it were at or after $x = 6$, the graph would be rising from 5.5 to 6, but $E(6) < E(5.5)$. So the peak lies between $x = 5$ and $x = 6$: between 500 m and 600 m in, and the estimate of its height is 25.3 m above the entrance. (Calculus puts it near $x \approx 5.63$, about 25.4 m up.)

(d) $E(8) = -\dfrac{1}{4}(8)(36)(1) = -72$: at the far end, 800 m in, the floor is 72 m below the entrance.

SCORING

1 point: zeros 0, 2, 7, with 2 a touch (double zero) and 7 the only place the floor passes below. 1 point: above on $0 < x < 2$ and $2 < x < 7$, below on $7 < x \le 8$, from a sign chart. 1 point: the turn count (at most

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3, two used by the peak in $0 < x < 2$ and the valley at 2), the three values, and the stretch from 500 m to 600 m with the estimate 25.3 m. 1 point: $E(8) = -72$, read as 72 m below the entrance at the far end.

12. Naomi counts visitors at a county fair that is open for 6 hours. The total number of visitors who have come in, in tens, is modeled by $V(t) = -t^3 + bt^2 + ct$ for $0 \le t \le 6$ hours after opening, where b and c are constants. Visitors were arriving fastest 3 hours after opening, at 112 tens of visitors per hour. (4 points)

(a) Find b and c.

(b) Find where V is concave up and where it is concave down, and its point of inflection. What does the point of inflection say about the fair?

(c) Were visitors still arriving at closing time? Explain.

ANSWER

(a) $b = 9$, $c = 85$. (b) Concave up on (0, 3), concave down on (3, 6); point of inflection (3, 309): 3,090 visitors had come in by the busiest moment. (c) Yes: the rate at $t = 6$ is 85 tens, or 850 visitors, per hour.

SOLUTION

(a) By the cubic result the rate at t is $-3t^2 + 2bt + c$, a quadratic that opens downward with its vertex at $t = \dfrac{2b}{6} = \dfrac{b}{3}$. Arrivals were fastest at its vertex, so $\dfrac{b}{3} = 3$ and $b = 9$. The largest rate is

$$-3(9) + 2(9)(3) + c = 27 + c = 112,$$

so $c = 85$. In vertex form the rate is $-3(t - 3)^2 + 112$.

(b) The rate increases on (0, 3) and decreases on (3, 6), so V is concave up on (0, 3) and concave down on (3, 6). The point of inflection is $(3, V(3))$, where $V(3) = -3^3 + 9(3)^2 + 85(3) = 309$. It marks the busiest moment of the day: 3,090 visitors had come in by then, and after it they still arrive, but more slowly.

(c) At $t = 6$ the rate is $-3(6 - 3)^2 + 112 = -3(9) + 112 = 85$, which is positive. Visitors were still arriving at 85 tens, or 850 visitors, per hour, so the total was still rising at closing.

SCORING

1 point for $b = 9$ (vertex of the rate at $t = 3$); 1 point for $c = 85$; 1 point for the concavity intervals and the point of inflection (3, 309) with its meaning; 1 point for part (c), with the rate 85 tens per hour as the reason.

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