# Precalculus Unit 1 worksheet: Concavity: when the rate of change itself changes (answer key)

**ANSWERS AT A GLANCE**

1. (a) concave up everywhere (b) concave down everywhere (c) neither (d) concave up everywhere (e) concave down on $(-\infty, 2)$, concave up on $(2, \infty)$ (f) concave down everywhere
2. (a) −4.8, −3.6, −2.6, −1.8, −1.2 degrees per minute. (b) Decreasing and concave up: the bread is cooling, but more slowly each step. (c) No: a drop of 8 degrees would be bigger than the last drop of 6, so the rates would decrease; concave up predicts a drop of less than 6, a temperature above 22 degrees.
3. (a) 0.2, 0.5, 0.8, 0.7, 0.4, 0.1 mm per minute. (b) Concave up through $t = 20$, concave down from $t = 30$; the concavity changes between $t = 20$ and $t = 30$, and the rain fell hardest then, at 0.8 mm per minute. (c) No: the step from $t = 40$ to $t = 60$ is twice as long, so its 8 mm is 0.4 mm per minute, less than the 0.7 before it. Raw differences compare rates only when the inputs are equally spaced.
4. (a) −15, −1, 1, 15. (b) The slopes increase, so $g$ looks concave up on both sides of $x = 3$; Omar is wrong. $(3, 1)$ is a turning point, the lowest point of the graph. (c) For example $y = x^3$, flat at its point of inflection $(0, 0)$.
5. (a) $P = (-2, -9)$; concave down on $(-\infty, -2)$, concave up on $(-2, \infty)$. (b) $f(-2 + t) + f(-2 - t) = -18$ for every $t$: the graph is symmetric about $P$ (a half turn about $P$ gives the same graph). (c) Both slopes are $\dfrac{f(-2 + s) - f(-2 + t)}{s - t}$. So the secant slopes on the left of $P$ are those on the right in reverse order: where they increase on the right, they decrease on the left.
6. See the proof below.

**WORKED SOLUTIONS**

**1.** Say where each function is concave up and where it is concave down, or say that it is neither. Name the result you used (quadratic, cubic, exponential or line), or reason from first differences if no result applies.

(a) $f(x) = 6(0.7)^x$ (b) $g(x) = 4x - 3x^2$ (c) $h(x) = 9 - 5x$

(d) $k(x) = \dfrac{1}{2}x^2 - 8x + 1$ (e) $p(x) = 2x^3 - 12x^2 + 7$ (f) $q(x) = -3(2)^x$

**ANSWER**

(a) concave up everywhere (b) concave down everywhere (c) neither (d) concave up everywhere (e) concave down on $(-\infty, 2)$, concave up on $(2, \infty)$ (f) concave down everywhere

**SOLUTION**

(a) Exponential result: $a = 6 > 0$ and $b = 0.7$, so the graph is concave up everywhere, even though it is falling.

(b) Quadratic result: $p = -3 < 0$, so concave down everywhere.

(c) A line: every secant slope is −5, so the slopes neither increase nor decrease. Neither.

(d) Quadratic result: $p = \dfrac{1}{2} > 0$, so concave up everywhere.

(e) Cubic result: $a = 2$, $b = -12$, so the point of inflection is at $x = -\dfrac{b}{3a} = -\dfrac{-12}{6} = 2$. With $a > 0$ the graph is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

(f) The exponential result needs $a > 0$, so work from first differences. Over $[x, x + d]$ the difference is $-3(2)^{x+d} + 3(2)^x = -3(2)^x(2^d - 1)$, which is negative and grows in size by the factor $2^d$ each step. The first differences decrease, so $q$ is concave down everywhere. (Its graph is the graph of $3(2)^x$, which is concave up, flipped over the $x$-axis.)

**2.** Jin takes a loaf of bread out of the oven and records the temperature at its center every 5 minutes.

| **minutes** $t$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| temperature (degrees Celsius) | 98 | 74 | 56 | 43 | 34 | 28 |


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(a) Find the average rate of change over each 5-minute step, in degrees per minute.

(b) Describe the temperature with "increasing" or "decreasing" and "concave up" or "concave down", and say in words what that means for the cooling.

(c) Jin predicts the bread will be at 20 degrees at $t = 30$. Is that prediction in line with the concavity of the table? Explain.

**ANSWER**

(a) −4.8, −3.6, −2.6, −1.8, −1.2 degrees per minute. (b) Decreasing and concave up: the bread is cooling, but more slowly each step. (c) No: a drop of 8 degrees would be bigger than the last drop of 6, so the rates would decrease; concave up predicts a drop of less than 6, a temperature above 22 degrees.

**SOLUTION**

(a) The first differences are 74 − 98 = −24, then −18, −13, −9 and −6. Divide each by the step of 5 minutes: −4.8, −3.6, −2.6, −1.8 and −1.2 degrees per minute.

(b) Every rate is negative, so the temperature is decreasing. The rates increase toward 0 (the second differences 6, 5, 4, 3 are positive), so the table is concave up. The bread cools fast at first and then more and more slowly.

(c) Jin's prediction is a drop of 28 − 20 = 8 degrees, a rate of −1.6 degrees per minute. That is less than the last rate, −1.2, so the rates would start to decrease and the table would turn concave down. If the cooling keeps slowing, the next drop should be less than 6 degrees, leaving the bread above 22 degrees at $t = 30$. The prediction is not in line with the table.

**3.** During a storm, Sofia reads the total rainfall in her rain gauge. She reads it every 10 minutes at first, then every 20 minutes.

| **minutes** $t$ | **0** | **10** | **20** | **30** | **40** | **60** | **80** |
|---|---|---|---|---|---|---|---|
| total rain (mm) | 0 | 2 | 7 | 15 | 22 | 30 | 32 |

(a) Find the average rate of rainfall over each step, in millimeters per minute.

(b) Where is the table concave up, where is it concave down, and between which readings does the concavity change? When was the rain falling hardest?

(c) Sofia looks at the first differences 2, 5, 8, 7, 8, 2 and says the rain picked up again after $t = 40$. Is she right? Explain.

**ANSWER**

(a) 0.2, 0.5, 0.8, 0.7, 0.4, 0.1 mm per minute. (b) Concave up through $t = 20$, concave down from $t = 30$; the concavity changes between $t = 20$ and $t = 30$, and the rain fell hardest then, at 0.8 mm per minute. (c) No: the step from $t = 40$ to $t = 60$ is twice as long, so its 8 mm is 0.4 mm per minute, less than the 0.7 before it. Raw differences compare rates only when the inputs are equally spaced.

**SOLUTION**

(a) Divide each change by the width of its step: $\frac{2}{10} = 0.2$, $\frac{5}{10} = 0.5$, $\frac{8}{10} = 0.8$, $\frac{7}{10} = 0.7$, then $\frac{8}{20} = 0.4$ and $\frac{2}{20} = 0.1$ mm per minute.

(b) At each inner reading compare the secant slope before it with the one after it. At $t = 10$, $0.2 < 0.5$, and at $t = 20$, $0.5 < 0.8$: the slopes increase, so the table is concave up through $t = 20$. At $t = 30$, $0.8 > 0.7$; at $t = 40$, $0.7 > 0.4$; at $t = 60$, $0.4 > 0.1$: the slopes decrease, so the table is concave down from $t = 30$. The concavity changes between $t = 20$ and $t = 30$, the step with the largest rate, 0.8 mm per minute: the rain fell hardest then and eased off after. The readings cannot place the change more closely.

(c) The difference of 8 mm from $t = 40$ to $t = 60$ covers 20 minutes, not 10, so the average rate there is 0.4 mm per minute, down from 0.7. The rain kept easing off. First differences stand in for rates only when the


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steps are equally wide.

**4.** Omar says that $g(x) = (x-3)^4 + 1$ has a point of inflection at $x = 3$, because its graph is flat there.

(a) Find the secant slopes on $[1, 2]$, $[2, 3]$, $[3, 4]$ and $[4, 5]$.

(b) What do the slopes suggest about the concavity of $g$, and is Omar right? What kind of point is (3, 1)?

(c) Give a function whose graph is flat at a point of inflection.

**ANSWER**

(a) −15, −1, 1, 15. (b) The slopes increase, so $g$ looks concave up on both sides of $x = 3$; Omar is wrong. (3, 1) is a turning point, the lowest point of the graph. (c) For example $y = x^3$, flat at its point of inflection (0, 0).

**SOLUTION**

(a) The values at $x = 1, 2, 3, 4, 5$ are 17, 2, 1, 2, 17. Each interval has width 1, so the slopes are the first differences: 2 − 17 = −15, 1 − 2 = −1, 2 − 1 = 1 and 17 − 2 = 15.

(b) The slopes increase all the way across, including from $[2, 3]$ to $[3, 4]$, so the concavity does not change at $x = 3$: $g$ is concave up there, as $x^4$ is. Being flat is not an inflection; an inflection needs the rate to switch between increasing and decreasing. The rate goes from negative to positive at $x = 3$, so (3, 1) is a turning point, the lowest point of the graph.

(c) $y = x^3$: by the cubic result ($a = 1$, $b = 0$) its point of inflection is at $x = -\dfrac{0}{3} = 0$, concave down to the left and concave up to the right. Its rate of change at $x$ is $3x^2$, which is 0 at $x = 0$, so the graph is flat at its point of inflection (0, 0). (Any $y = (x-h)^3 + k$ works.)

**5.** Let $f(x) = x^3 + 6x^2 + 9x - 7$.

(a) Find the point of inflection $P$ of $f$, and the intervals where $f$ is concave up and concave down.

(b) Show that $f(-2+t) = -9 - 3t + t^3$ for every number $t$. Use it to show that $f(-2+t) + f(-2-t)$ is the same number for every $t$, and say what that means for the graph and $P$.

(c) Use (b) to show that $m(-2-s, -2-t) = m(-2+t, -2+s)$ whenever $0 < t < s$. Explain why this makes the concavity to the left of $P$ the opposite of the concavity to the right.

**ANSWER**

(a) $P = (-2, -9)$; concave down on $(-\infty, -2)$, concave up on $(-2, \infty)$. (b) $f(-2+t) + f(-2-t) = -18$ for every $t$: the graph is symmetric about $P$ (a half turn about $P$ gives the same graph). (c) Both slopes are $\dfrac{f(-2+s) - f(-2+t)}{s-t}$. So the secant slopes on the left of $P$ are those on the right in reverse order: where they increase on the right, they decrease on the left.

**SOLUTION**

(a) By the cubic result with $a = 1$ and $b = 6$, the point of inflection is at $x = -\dfrac{b}{3a} = -\dfrac{6}{3} = -2$. Since $a > 0$, $f$ is concave down on $(-\infty, -2)$ and concave up on $(-2, \infty)$. The point is $P = (-2, f(-2)) = (-2, -9)$.

(b) Expand $f(-2+t) = (-2+t)^3 + 6(-2+t)^2 + 9(-2+t) - 7$. The $t^3$ term is $t^3$. The $t^2$ terms are $-6t^2$ and $6t^2$, which cancel. The $t$ terms are $12t$, $-24t$ and $9t$, which add to $-3t$. The constant terms add to $f(-2) = -9$. So $f(-2+t) = -9 - 3t + t^3$.

Replacing $t$ by $-t$ gives $f(-2-t) = -9 + 3t - t^3$. Adding, the $t$ and $t^3$ terms cancel:

$$f(-2+t) + f(-2-t) = 2(-9) = -18.$$

So the points of the graph at $x = -2+t$ and $x = -2-t$ are always equally far above and below $P$: $P$ is


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the midpoint of the segment joining them. The graph is symmetric about $P$; a half turn about $P$ maps it onto itself.

(c) By (b), $f(-2-t) = -18 - f(-2+t)$ and $f(-2-s) = -18 - f(-2+s)$. So

$$m(-2-s,-2-t) = \frac{f(-2-t)-f(-2-s)}{s-t} = \frac{f(-2+s)-f(-2+t)}{s-t} = m(-2+t,-2+s).$$

Take three inputs to the right of $P$, $-2+t_1 < -2+t_2 < -2+t_3$. Their mirror images $-2-t_3 < -2-t_2 < -2-t_1$ lie to the left, and reading left to right the two slopes there are $m(-2+t_2,-2+t_3)$, then $m(-2+t_1,-2+t_2)$: the slopes on the right in reverse order. Concave up on the right means $m(-2+t_1,-2+t_2) < m(-2+t_2,-2+t_3)$, so on the left the slopes decrease: concave down. The half turn changes a cup into a cap, which is why the concavity switches at $P$.

**6.** Let $f(x) = x^3$.

(a) Show that the secant slope from $a$ to $b$ is $m(a,b) = a^2 + ab + b^2$.

(b) Prove that $f$ is concave up on $[0,\infty)$: for every $0 \le a < b < c$, show $m(a,b) < m(b,c)$.

(c) Prove in the same way that $f$ is concave down on $(-\infty, 0]$, and name the point of inflection.

**ANSWER**

(a) $m(a,b) = \dfrac{b^3-a^3}{b-a} = a^2+ab+b^2$. (b) $m(b,c) - m(a,b) = (c-a)(a+b+c) > 0$. (c) For $a < b < c \le 0$, $a+b+c < 0$, so the difference is negative; the point of inflection is $(0,\,0)$.

**SOLUTION**

(a) Factor the difference of cubes, $b^3 - a^3 = (b-a)(a^2+ab+b^2)$, so for $a \ne b$

$$m(a,b) = \frac{b^3-a^3}{b-a} = a^2+ab+b^2.$$

(b) Take $0 \le a < b < c$. By (a),

$$\begin{aligned} m(b,c) - m(a,b) &= (b^2+bc+c^2) - (a^2+ab+b^2) \\ &= (c^2-a^2) + b(c-a) \\ &= (c-a)(a+b+c). \end{aligned}$$

Here $c-a > 0$, and $a+b+c > 0$ because $b$ and $c$ are positive and $a \ge 0$. So $m(a,b) < m(b,c)$, which is the definition of concave up on $[0,\infty)$.

(c) Take $a < b < c \le 0$. The same difference is $(c-a)(a+b+c)$, with $c-a > 0$ and $a+b+c < 0$ because $a$ and $b$ are negative and $c \le 0$. So $m(a,b) > m(b,c)$: concave down on $(-\infty, 0]$. The concavity changes at $x = 0$ and $f$ is continuous there, so $(0, f(0)) = (0,0)$ is the point of inflection.

**SCORING**

1 point for (a), the difference of cubes factored and divided by $b-a$. 2 points for (b): 1 for simplifying $m(b,c) - m(a,b)$ to $(c-a)(a+b+c)$, 1 for the sign of each factor and the conclusion from the definition. 2 points for (c): 1 for the signs with $a < b < c \le 0$ and the conclusion concave down, 1 for the point of inflection $(0,\,0)$ with the reason (concavity changes there and $f$ is continuous).


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