Precalculus Unit 1 worksheet: Degree, end behavior and the shape of a polynomial graph (answer key)

ANSWERS AT A GLANCE

1. (a) $-6x^3$; as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$ (b) $-8x^4$; as $x \to \pm\infty$, $p(x) \to -\infty$ (c) $12x^2$; as $x \to \pm\infty$, $p(x) \to \infty$ (d) $\frac{1}{2}x^5$; as $x \to -\infty$, $p(x) \to -\infty$, and as $x \to \infty$, $p(x) \to \infty$
2. (a) degree 5, leading term $-4x^5$ (b) as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$ (c) $(0, 144)$ (d) 4
3. (a) impossible (b) possible, for example $p(x) = x^3$ (c) impossible (d) possible, for example $p(x) = -x^6 - 1$
4. (a) at most 2; $x = -2 + 2\sqrt{2}$ and $x = -2 - 2\sqrt{2}$ (b) at most 2
5. A is (4), B is (1), C is (2), D is (3).
6. (a) 5 and 7 are possible; 3, 4 and 6 are not. (b) negative (c) degree at least 3; the 4 turning points force degree at least 5.

WORKED SOLUTIONS

1. Collect like terms where needed, then give the leading term of each polynomial and describe its end behavior in arrow notation.

(a) $p(x) = 7 + 2x^2 - 6x^3$ (b) $p(x) = x^3(4 - x)^2 - x^5$ (c) $p(x) = (2x + 1)^3 - 8x^3$

(d) $p(x) = 9 - x^4 + \frac{1}{2}x^5$

ANSWER

(a) $-6x^3$; as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$ (b) $-8x^4$; as $x \to \pm\infty$, $p(x) \to -\infty$ (c) $12x^2$; as $x \to \pm\infty$, $p(x) \to \infty$ (d) $\frac{1}{2}x^5$; as $x \to -\infty$, $p(x) \to -\infty$, and as $x \to \infty$, $p(x) \to \infty$

SOLUTION

(a) The highest power is x^3, written last: the leading term is $-6x^3$. Odd degree, negative leading coefficient: the left end rises and the right end falls.

(b) Expand first: $x^3(16 - 8x + x^2) - x^5 = 16x^3 - 8x^4 + x^5 - x^5 = -8x^4 + 16x^3$. The x^5 terms cancel, so the leading term is $-8x^4$, not anything in x^5. Even degree, negative leading coefficient: both ends fall.

(c) $(2x + 1)^3 = 8x^3 + 12x^2 + 6x + 1$, so $p(x) = 12x^2 + 6x + 1$. The cubes cancel and the leading term is $12x^2$. Even degree, positive leading coefficient: both ends rise.

(d) The leading term is $\frac{1}{2}x^5$. Odd degree, positive leading coefficient: the left end falls and the right end rises.

2. Let $p(x) = 4(x - 3)^2(4 - x)(x^2 + 1)$.

(a) Find the degree and the leading term of p.

(b) Describe the end behavior of p in arrow notation.

(c) Find the y-intercept of the graph of p.

(d) At most how many turning points can the graph of p have?

ANSWER

(a) degree 5, leading term $-4x^5$ (b) as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$ (c) $(0, 144)$ (d) 4

SOLUTION

(a) The leading term of a product is the product of the factors' leading terms. The leading terms are 4, x^2, $-x$ and x^2, so the leading term of p is $4 \cdot x^2 \cdot (-x) \cdot x^2 = -4x^5$, of degree 2 + 1 + 2 = 5.

(b) The degree 5 is odd and the leading coefficient is negative, so as $x \to -\infty$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$.

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(c) $p(0) = 4 \cdot (-3)^2 \cdot 4 \cdot 1 = 144$, so the y-intercept is $(0, 144)$.

(d) A polynomial of degree 5 has at most 5 − 1 = 4 turning points.

3. Decide whether a polynomial with the description exists. If it does, write one. If it does not, say which fact about polynomials rules it out.

(a) Degree 5, with 6 real zeros.

(b) Degree 3, with no turning points.

(c) Degree 4, with every real number as an output.

(d) Degree 6, with no real zeros and both ends falling.

ANSWER

(a) impossible (b) possible, for example $p(x) = x^3$ (c) impossible (d) possible, for example $p(x) = -x^6 - 1$

SOLUTION

(a) Impossible: a polynomial of degree 5 has at most 5 real zeros.

(b) Possible: $p(x) = x^3$ rises everywhere, since a larger input always has a larger cube, so it never switches between rising and falling.

(c) Impossible: a polynomial of even degree has a range bounded on one side, $y \geq m$ or $y \leq M$, so some real numbers are never outputs.

(d) Possible: $p(x) = -x^6 - 1$. Its leading coefficient is negative and the degree is even, so both ends fall, and since $x^6 \geq 0$, $p(x) \leq -1$ for every x: it is never 0.

4. A point where two graphs meet is a solution of the equation that sets the two rules equal.

(a) At most how many points can the graphs of $y = 2x^5 - x^2 + 3$ and $y = 2x^5 + 4x - 1$ have in common? Find the x-coordinate of each one.

(b) Two different cubic polynomials f and g have the same leading coefficient. At most how many points can their graphs share? Explain.

ANSWER

(a) at most 2; $x = -2 + 2\sqrt{2}$ and $x = -2 - 2\sqrt{2}$ (b) at most 2

SOLUTION

(a) Set the rules equal: $2x^5 - x^2 + 3 = 2x^5 + 4x - 1$. The x^5 terms cancel, leaving $-x^2 - 4x + 4 = 0$, a polynomial equation of degree 2, with at most 2 real solutions. Multiply by −1: $x^2 + 4x - 4 = 0$, so

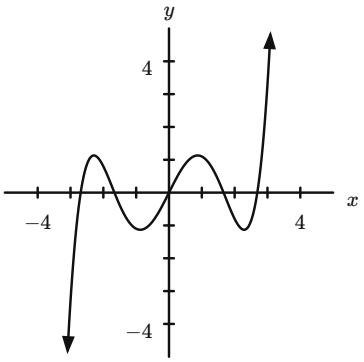
$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} = \frac{-4 \pm 4\sqrt{2}}{2} = -2 \pm 2\sqrt{2}.$$

Both are real, so the graphs share exactly 2 points, at $x = -2 + 2\sqrt{2}$ and $x = -2 - 2\sqrt{2}$.

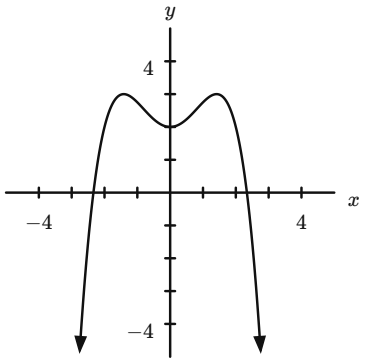
(b) The graphs meet where $f(x) - g(x) = 0$. The leading terms are equal, so they cancel, and $f - g$ has degree at most 2. It is not the zero polynomial, since f and g are different. If $f - g$ is a nonzero constant the graphs never meet; otherwise it has degree 1 or 2 and at most 2 real zeros. So the graphs share at most 2 points. Two is reached: $y = x^3 + x^2$ and $y = x^3 + 1$ meet where $x^2 - 1 = 0$, at $x = -1$ and $x = 1$.

5. Each graph A to D is the graph of one of these polynomials. Match each graph with its polynomial, and give a reason for each match that uses the end behavior and the number of turning points.

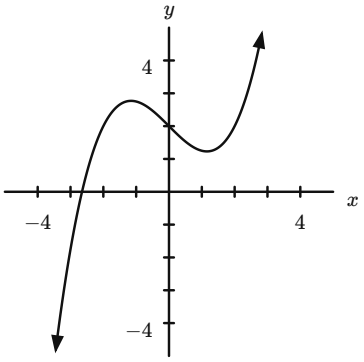
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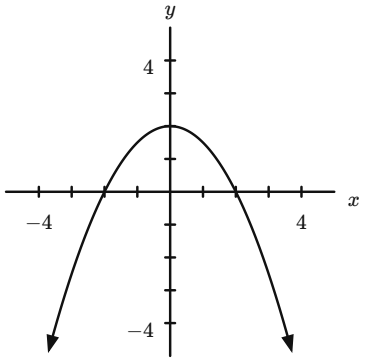
A



B



C



D

(1) $y = 2 + x^2 - \frac{1}{4}x^4$ (2) $y = \frac{1}{4}x^3 - x + 2$ (3) $y = 2 - \frac{1}{2}x^2$ (4) $y = \frac{1}{10}x^5 - x^3 + 2x$

ANSWER

A is (4), B is (1), C is (2), D is (3).

SOLUTION

Sort by the ends first. In A and C the left end falls and the right end rises: odd degree, positive leading coefficient. Of the four polynomials, (2) (leading term $\frac{1}{4}x^3$) and (4) (leading term $\frac{1}{10}x^5$) fit. In B and D both ends fall: even degree, negative leading coefficient, which fits (1) (leading term $-\frac{1}{4}x^4$) and (3) (leading term $-\frac{1}{2}x^2$).

Then count turns. A has 4 turning points, two peaks and two valleys. A degree 3 polynomial has at most 2, so A is not (2): A is (4) and C is (2). B has 3 turning points, and a degree 2 polynomial has at most 1, so B is (1) and D, a single hill, is (3).

Check: (3) has its top at (0, 2), and (1) has a valley at (0, 2) on the y-axis, as D and B show.

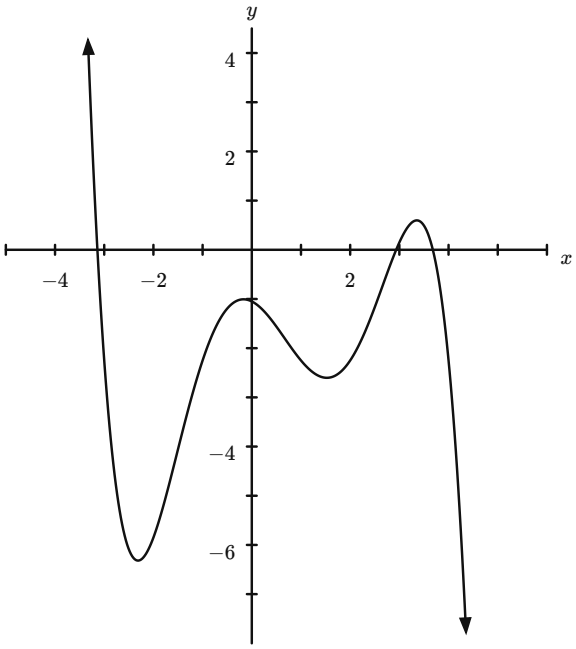
6. The graph of a polynomial p is shown. Apart from its two ends, which go on in the directions of the arrows, the whole graph is in view.

(a) For each of the degrees 3, 4, 5, 6 and 7, say whether p could have that degree, and give the reason.

(b) What is the sign of the leading coefficient of p?

(c) Counting the x-intercepts alone, what least degree would you get? Explain why the turning points give a stronger bound for this graph.

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ANSWER

(a) 5 and 7 are possible; 3, 4 and 6 are not. (b) negative (c) degree at least 3; the 4 turning points force degree at least 5.

SOLUTION

The left end rises and the right end falls. The graph has 4 turning points (valley, peak, valley, peak from left to right) and 3 x-intercepts.

(a) Degree 4 and degree 6 are even, and an even degree makes both ends go the same way; here they go opposite ways, so the degree is odd. Degree 3 allows at most 2 turning points, and the graph has 4. Degree 5 allows 4 and degree 7 allows 6, and both are odd, so 5 and 7 are possible.

(b) Odd degree with the left end rising and the right end falling: the leading coefficient is negative.

(c) A polynomial of degree n has at most n real zeros, so three x-intercepts need $n \geq 3$: the intercepts alone give degree at least 3. The turns need $n - 1 \geq 4$, so $n \geq 5$. The intercepts say less here because the low peak just left of the y-axis never reaches the axis: the graph turns there without crossing.

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