Precalculus Unit 1 worksheet: The difference quotient (answer key)

ANSWERS AT A GLANCE

1. (a) $\dfrac{2x+h-6}{4}$ (b) $5-4x-2h$ (c) $4x+2h-5$ (d) $3x^2+3xh+h^2-6x-3h$
2. The difference quotient is $-4x-2h+4$; the secant slope is 4; the secant line is $y=4x-7$.
3. (a) $4t+2h-100$ liters per minute (b) −80 liters per minute (c) from $t=9$ to $t=11$ minutes
4. (a) $\dfrac{-2}{\sqrt{9-2x-2h}+\sqrt{9-2x}}$, for $x\le\dfrac{9}{2}$ and $x+h\le\dfrac{9}{2}$, not both equal to $\dfrac{9}{2}$. (b) $-\dfrac{1}{4}$
5. $-\dfrac{2x+h}{x^2(x+h)^2}$, for $x\neq 0$ and $x+h\neq 0$.
6. (a) $-\dfrac{6}{x(x+h)}$, for $x\neq 0$ and $x+h\neq 0$. (b) On one branch x and $x+h$ have the same sign, so $x(x+h)>0$ and the quotient is negative. (c) 1; the ends lie on different branches, so the quotient describes no single stretch of the graph.

WORKED SOLUTIONS

1. Find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ of each function.

(a) $f(x)=\dfrac{x^2-6x}{4}$ (b) $f(x)=5x-2x^2$ (c) $f(x)=(2x+1)(x-3)$

(d) $f(x)=x^2(x-3)$

ANSWER

(a) $\dfrac{2x+h-6}{4}$ (b) $5-4x-2h$ (c) $4x+2h-5$ (d) $3x^2+3xh+h^2-6x-3h$

SOLUTION

(a) $f(x+h)-f(x)=\dfrac{(x+h)^2-6(x+h)-x^2+6x}{4}=\dfrac{2xh+h^2-6h}{4}$. Dividing by h gives $\dfrac{2x+h-6}{4}$.

(b) $f(x+h)=5x+5h-2x^2-4xh-2h^2$. Subtracting $f(x)=5x-2x^2$ leaves $5h-4xh-2h^2$; dividing by h gives $5-4x-2h$.

(c) Multiply out first: $f(x)=2x^2-5x-3$. Then $f(x+h)=2x^2+4xh+2h^2-5x-5h-3$, and subtracting $f(x)$ leaves $4xh+2h^2-5h$. Dividing by h gives $4x+2h-5$.

(d) Multiply out first: $f(x)=x^3-3x^2$. With $(x+h)^3=x^3+3x^2h+3xh^2+h^3$,

$$f(x+h)=x^3+3x^2h+3xh^2+h^3-3x^2-6xh-3h^2.$$

Subtracting $f(x)$ leaves $3x^2h+3xh^2+h^3-6xh-3h^2$; dividing by h gives $3x^2+3xh+h^2-6x-3h$. Check with $x=1$, $h=1$: $f(1)=-2$ and $f(2)=-4$, average rate −2; the formula gives $3+3+1-6-3=-2$.

Each holds for $h\neq 0$.

2. Let $f(x)=-2x^2+4x-5$. Find and simplify the difference quotient of f. Use it to find the slope of the secant line through the points of the graph at $x=-1$ and $x=1$, then write an equation of that secant line in slope-intercept form.

ANSWER

The difference quotient is $-4x-2h+4$; the secant slope is 4; the secant line is $y=4x-7$.

SOLUTION

$f(x+h)-f(x)=-4xh-2h^2+4h$, since the terms without h cancel. Dividing by $h\neq 0$:

$$\frac{f(x+h)-f(x)}{h}=-4x-2h+4.$$

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The secant runs from $x = -1$ to $x = 1$, so put $x = -1$ and $h = 2$: the slope is $-4 \cdot (-1) - 2 \cdot 2 + 4 = 4$. The line passes through $(-1, f(-1)) = (-1, -11)$, so $y - (-11) = 4(x - (-1))$, which is $y = 4x - 7$. Check: $f(1) = -3$, and $\dfrac{-3 - (-11)}{2} = 4$. The line also passes through $(1, -3)$.

3. Diego drains a hot tub. The volume of water left t minutes after the drain opens is $V(t) = 2t^2 - 100t + 1{,}250$ liters, for $0 \le t \le 25$.

(a) Find and simplify $\dfrac{V(t+h) - V(t)}{h}$, and give its units.

(b) Use it to find the average rate of change of the volume over the first 10 minutes.

(c) During which 2-minute stretch does the water drain at an average of 60 liters per minute?

ANSWER

(a) $4t + 2h - 100$ liters per minute (b) -80 liters per minute (c) from $t = 9$ to $t = 11$ minutes

SOLUTION

(a) $V(t+h) = 2(t+h)^2 - 100(t+h) + 1{,}250 = 2t^2 + 4th + 2h^2 - 100t - 100h + 1{,}250$. Subtracting $V(t)$ leaves $4th + 2h^2 - 100h$, and dividing by $h \neq 0$:

$$\frac{V(t+h) - V(t)}{h} = 4t + 2h - 100$$

liters per minute, the average rate of change of the volume from t to $t + h$.

(b) The first 10 minutes start at $t = 0$ with $h = 10$: $4(0) + 2(10) - 100 = -80$ liters per minute. The water leaves at an average of 80 liters per minute; the minus sign means the volume falls.

(c) Draining at 60 liters per minute means a rate of change of -60. With $h = 2$:

$$4t + 2(2) - 100 = -60,$$

so $4t = 100 - 60 - 2(2)$ and $t = 9$. The stretch runs from $t = 9$ to $t = 11$ minutes. Check: $\dfrac{V(11) - V(9)}{2} = -60$.

4. Let $f(x) = \sqrt{9 - 2x}$.

(a) Find and simplify the difference quotient of f, and say which values of x and $x + h$ it needs.

(b) Use it to find the average rate of change of f on $[-8, 0]$. Check your answer from $f(-8)$ and $f(0)$.

ANSWER

(a) $\dfrac{-2}{\sqrt{9 - 2x - 2h} + \sqrt{9 - 2x}}$, for $x \le \dfrac{9}{2}$ and $x + h \le \dfrac{9}{2}$, not both equal to $\dfrac{9}{2}$. (b) $-\dfrac{1}{4}$

SOLUTION

(a) The quotient is $\dfrac{\sqrt{9 - 2x - 2h} - \sqrt{9 - 2x}}{h}$. Multiply top and bottom by the conjugate $\sqrt{9 - 2x - 2h} + \sqrt{9 - 2x}$. The numerator becomes

$$(9 - 2x - 2h) - (9 - 2x) = -2h,$$

and canceling h:

$$\frac{f(x+h) - f(x)}{h} = \frac{-2}{\sqrt{9 - 2x - 2h} + \sqrt{9 - 2x}}.$$

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Both ends must be in the domain, $9-2x \geq 0$ and $9-2x-2h \geq 0$, so $x \leq \frac{9}{2}$ and $x+h \leq \frac{9}{2}$, and they may not both be $\frac{9}{2}$, or the conjugate would be 0.

(b) The interval starts at $x=-8$ and has width $h=8$: $\frac{-2}{\sqrt{9}+\sqrt{25}} = \frac{-2}{8} = -\frac{1}{4}$.

Check: $f(-8)=5$ and $f(0)=3$, so $\frac{3-5}{8} = -\frac{1}{4}$.

5. Find and simplify the difference quotient of $f(x) = \frac{1}{x^2}$, and say which values of x and $x+h$ it needs. Check your answer with $x=1$ and $h=1$.

ANSWER

$-\frac{2x+h}{x^2(x+h)^2}$, for $x \neq 0$ and $x+h \neq 0$.

SOLUTION

$f(x+h)-f(x) = \frac{1}{(x+h)^2} - \frac{1}{x^2}$. Over the common denominator $x^2(x+h)^2$ the numerator is

$$x^2-(x+h)^2 = x^2-x^2-2xh-h^2 = -2xh-h^2 = -h(2x+h).$$

Dividing by $h \neq 0$ cancels the h:

$$\frac{f(x+h)-f(x)}{h} = -\frac{2x+h}{x^2(x+h)^2}.$$

Both ends must be in the domain: $x \neq 0$ and $x+h \neq 0$.

Check: $f(1)=1$ and $f(2)=\frac{1}{4}$, so the average rate on $[1,2]$ is $\frac{1}{4}-1 = -\frac{3}{4}$. The formula gives $-\frac{3}{1\cdot 4} = -\frac{3}{4}$.

6. Let $f(x) = \frac{6}{x}$.

(a) Find and simplify the difference quotient of f.

(b) Explain why every secant line joining two points on the same branch of the graph has a negative slope.

(c) Evaluate your quotient at $x=-2$ and $h=5$. Check the number from $f(-2)$ and $f(3)$, then explain why it does not describe how the graph behaves between $x=-2$ and $x=3$.

ANSWER

(a) $-\frac{6}{x(x+h)}$, for $x \neq 0$ and $x+h \neq 0$. (b) On one branch x and $x+h$ have the same sign, so $x(x+h) > 0$ and the quotient is negative. (c) 1; the ends lie on different branches, so the quotient describes no single stretch of the graph.

SOLUTION

(a) Over the common denominator $x(x+h)$,

$$f(x+h)-f(x) = \frac{6x-6(x+h)}{x(x+h)} = \frac{-6h}{x(x+h)},$$

and dividing by $h \neq 0$ gives $\frac{f(x+h)-f(x)}{h} = -\frac{6}{x(x+h)}$, for $x \neq 0$ and $x+h \neq 0$.

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(b) The difference quotient is the slope of the secant through $(x, f(x))$ and $(x+h, f(x+h))$. On one branch both inputs are positive or both are negative, so $x(x+h) > 0$, and $-\dfrac{6}{x(x+h)}$ is negative.

(c) $-\dfrac{6}{(-2)(3)} = 1$. Directly: $f(-2) = -3$ and $f(3) = 2$, so $\dfrac{2-(-3)}{5} = 1$. But 0 lies between the ends and is not in the domain: the two ends are on different branches, and the graph falls on each branch. The positive number is the slope of a line joining the branches, not a rate of any single stretch of the graph.

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