Precalculus Unit 1 worksheet: Rates at an instant from shrinking intervals (answer key)

ANSWERS AT A GLANCE

1. (a) $9 + 6h + h^2$. (b) 9. (c) The right estimates are off by 0.61, 0.0601 and 0.006001; the left estimates by 0.59, 0.0599 and 0.005999. Both are within 0.1 for 0.01 and 0.001.
2. (a) 9; for a quadratic the centered average is the same for every width, so it is the rate. (b) $y = 9x - 5$, and $f(-0.9) \approx -13.1$. (c) $f(-0.9) = -13.12$; the estimate misses by 0.02.
3. (a) 2.1 and 2.01. (b) 11.9, 101.99 and 1001.999. (c) No. The right estimates approach 2, but the left estimates grow without bound and approach no number, so f has no rate of change at 1.
4. (a) Right 0.36, left 0.38, centered 0.37 degrees Celsius per second. (b) 0.37; best estimate about 0.37 degrees Celsius per second. (c) There is no reading at $t = 30$; with $h = 20$ the centered average is 0.39 degrees Celsius per second.
5. (a) Right: 10 (width 0.01) and 8 (width 0.1); left: 8; centered: 8 meters per second. (b) Up to 10, 1, 1 and 0.5 meters per second. (c) The centered average: about 8 meters per second.
6. (a) Shown in the solution. (b) The rate at 3 is 6.9; the coefficient of x^2 is 2. (c) −5.1.

WORKED SOLUTIONS

1. Let $f(x) = x^3 - 3x$.

(a) Simplify $\dfrac{f(2+h) - f(2)}{h}$ for $h \neq 0$.

(b) Use your answer to (a) to find the rate of change of f at $x = 2$. Explain.

(c) For each width 0.1, 0.01 and 0.001, find how far the right estimate and the left estimate are from the rate. For which of these widths are both estimates within 0.1 of the rate?

ANSWER

(a) $9 + 6h + h^2$. (b) 9. (c) The right estimates are off by 0.61, 0.0601 and 0.006001; the left estimates by 0.59, 0.0599 and 0.005999. Both are within 0.1 for 0.01 and 0.001.

SOLUTION

(a) Expanding $(2+h)^3 = 2^3 + 12h + 6h^2 + h^3$,

$$f(2+h) - f(2) = 12h + 6h^2 + h^3 - 3h,$$

so for $h \neq 0$ the quotient is $9 + 6h + h^2$.

(b) As h shrinks toward 0 from either side, $6h + h^2$ shrinks toward 0, so the right and the left estimates both approach 9. The rate of change at $x = 2$ is 9.

(c) The right estimate of width w uses $h = w$ and is above the rate by $6w + w^2$: by 0.61, 0.0601 and 0.006001 for $w = 0.1, 0.01, 0.001$. The left estimate uses $h = -w$ and is below the rate by $6w - w^2$: by 0.59, 0.0599 and 0.005999. Both errors are at most 0.1 for 0.01 and 0.001.

2. Let $f(x) = -2x^2 + 5x - 7$.

(a) Find the centered average of f at $x = -1$ with the wide width $h = 2$, and explain why this one number is already the rate of change of f at −1.

(b) Write the tangent line at $x = -1$ in the form $y = mx + b$, and use it to estimate $f(-0.9)$.

(c) Find $f(-0.9)$ exactly. By how much does the estimate in (b) miss?

ANSWER

(a) 9; for a quadratic the centered average is the same for every width, so it is the rate. (b) $y = 9x - 5$, and $f(-0.9) \approx -13.1$. (c) $f(-0.9) = -13.12$; the estimate misses by 0.02.

SOLUTION

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(a) $f(1) = -4$ and $f(-3) = -40$, so the centered average is

$$\frac{f(1) - f(-3)}{2 \cdot 2} = \frac{36}{4} = 9.$$

For a quadratic $px^2 + qx + c$ the centered average at a equals $2pa + q$ for every $h > 0$, here $2 \cdot (-2) \cdot (-1) + 5 = 9$. The width does not matter, so shrinking it would change nothing: the rate at -1 is 9.

(b) $f(-1) = -14$, so the tangent line is $y = -14 + 9(x + 1)$, which is $y = 9x - 5$. At $x = -0.9$ it gives $-14 + 9(0.1) = -13.1$.

(c) Substituting, $f(-0.9) = -13.12$. The estimate misses by 0.02. In general $f(a + h) = f(a) + (2pa + q)h + ph^2$, so the tangent line misses a quadratic by $|p|h^2$, here 0.02 with $h = 0.1$.

3. Let

$$f(x) = \begin{cases} x^2, & x < 1 \\ x^2 + 1, & x \geq 1. \end{cases}$$

(a) Find the right estimates $\dfrac{f(1 + h) - f(1)}{h}$ at $x = 1$ with $h = 0.1$ and $h = 0.01$.

(b) Find the left estimates at $x = 1$ with $h = -0.1$, $h = -0.01$ and $h = -0.001$.

(c) Ben looks only at part (a) and says the rate of change of f at 1 is 2. Is he right? Explain, using the definition of the rate at an instant.

ANSWER

(a) 2.1 and 2.01. (b) 11.9, 101.99 and 1001.999. (c) No. The right estimates approach 2, but the left estimates grow without bound and approach no number, so f has no rate of change at 1.

SOLUTION

Here $f(1) = 1^2 + 1 = 2$.

(a) $f(1.1) = 1.21 + 1 = 2.21$, so $\dfrac{2.21 - 2}{0.1} = 2.1$. $f(1.01) = 2.0201$, so $\dfrac{2.0201 - 2}{0.01} = 2.01$.

(b) To the left of 1 the rule is x^2. $f(0.9) = 0.81$: $\dfrac{0.81 - 2}{-0.1} = 11.9$. $f(0.99) = 0.9801$: $\dfrac{0.9801 - 2}{-0.01} = 101.99$. $f(0.999) = 0.998001$: $\dfrac{0.998001 - 2}{-0.001} = 1001.999$.

(c) No. A rate at 1 needs the right estimates and the left estimates to approach the same number. The right ones approach 2, but the left ones grow without bound: for $h < 0$ the quotient is $2 + h - \dfrac{1}{h}$, and $-\dfrac{1}{h}$ grows huge as h shrinks. The graph jumps up by 1 at $x = 1$, so every left secant through (1, 2) climbs steeply to reach it. f has no rate of change at 1.

4. Maya heats water in an electric kettle and reads a thermometer at the times below. She skipped some readings, so the times are not evenly spaced.

t **(seconds)**	0	20	40	50	60	80
temperature (degrees Celsius)	22.0	31.0	39.2	43.0	46.6	53.2

(a) At $t = 50$, find a right estimate, a left estimate and a centered average of the rate at which the temperature rises, each with the narrowest width the table allows.

(b) Find the centered average at $t = 50$ with $h = 30$. What is your best estimate of the rate at $t = 50$?

(c) Explain why the table cannot give a centered average at $t = 40$ with $h = 10$. Find the narrowest centered average it does allow at $t = 40$.

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ANSWER

(a) Right 0.36, left 0.38, centered 0.37 degrees Celsius per second. (b) 0.37; best estimate about 0.37 degrees Celsius per second. (c) There is no reading at $t = 30$; with $h = 20$ the centered average is 0.39 degrees Celsius per second.

SOLUTION

(a) The nearest readings to $t = 50$ are 10 seconds away on each side, so $h = 10$.

Right: $\dfrac{46.6 - 43.0}{10} = 0.36$. Left: $\dfrac{43.0 - 39.2}{10} = 0.38$. Centered: $\dfrac{46.6 - 39.2}{20} = 0.37$, the mean of the two.

(b) $\dfrac{53.2 - 31.0}{60} = \dfrac{22.2}{60} = 0.37$. Two centered averages of different widths agree, so the best estimate is about 0.37 degrees Celsius per second.

(c) A centered interval of width 20 around 40 runs from 30 to 50, and there is no reading at $t = 30$. The readings that sit the same distance on both sides of 40 are at 20 and 60, so $h = 20$: $\dfrac{46.6 - 31.0}{40} = \dfrac{15.6}{40} = 0.39$ degrees Celsius per second.

5. Diego rides a sprint on a velodrome. A laser timer records his distance from the start line, rounded to the nearest 0.1 meter.

t (seconds)	3.90	4.00	4.01	4.10
distance (meters)	29.4	30.2	30.3	31.0

(a) At $t = 4$, find the right estimates with widths 0.01 and 0.1, the left estimate with width 0.1, and the centered average with $h = 0.1$.

(b) Each reading can be off by up to 0.05 meter, so the difference of two readings can be off by up to 0.1 meter. At most, how far off can each of your four estimates be because of the rounding?

(c) Which estimate should Diego trust most, and about how fast was he riding at $t = 4$?

ANSWER

(a) Right: 10 (width 0.01) and 8 (width 0.1); left: 8; centered: 8 meters per second. (b) Up to 10, 1, 1 and 0.5 meters per second. (c) The centered average: about 8 meters per second.

SOLUTION

(a) Right, width 0.01: $\dfrac{30.3 - 30.2}{0.01} = 10$. Right, width 0.1: $\dfrac{31.0 - 30.2}{0.1} = 8$. Left, width 0.1: $\dfrac{30.2 - 29.4}{0.1} = 8$. Centered, $h = 0.1$: $\dfrac{31.0 - 29.4}{0.2} = 8$ meters per second.

(b) An error of up to 0.1 meter in the difference, divided by the width, gives the largest error in the rate: $\dfrac{0.1}{0.01} = 10$, $\dfrac{0.1}{0.1} = 1$, $\dfrac{0.1}{0.1} = 1$ and $\dfrac{0.1}{0.2} = 0.5$ meters per second.

(c) The narrowest interval gives the least trustworthy number here: its rounding error could be as large as the whole speed. The centered average has the smallest possible rounding error and also cancels most of the left and right errors, so Diego's speed was about 8 meters per second, somewhere between 7.5 and 8.5 from rounding alone.

6. (a) Let $f(x) = px^2 + qx + c$ and $h > 0$. Show that the right estimate $\dfrac{f(a+h) - f(a)}{h}$ is $2pa + q + ph$ and the left estimate $\dfrac{f(a) - f(a-h)}{h}$ is $2pa + q - ph$.

(b) Leah knows only that g is a quadratic. At $x = 3$, with width 0.2, her right estimate of the rate is 7.3 and her left estimate is 6.5. Find the rate of change of g at 3 and the coefficient of x^2 in g.

(c) Find the coefficient of x in g.

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ANSWER

(a) Shown in the solution. (b) The rate at 3 is 6.9; the coefficient of x^2 is 2. (c) −5.1.

SOLUTION

(a) $f(a+h) - f(a) = p\big((a+h)^2 - a^2\big) + qh = p(2ah + h^2) + qh$. Dividing by h gives $2pa + ph + q$, that is $2pa + q + ph$.

$f(a) - f(a-h) = p\big(a^2 - (a-h)^2\big) + qh = p(2ah - h^2) + qh$. Dividing by h gives $2pa + q - ph$.

(b) The two estimates are $2pa + q$ plus and minus ph, so their mean is the rate, $2pa + q$: $\dfrac{7.3 + 6.5}{2} = 6.9$. (This mean is also the centered average with $h = 0.2$, which for a quadratic is exact.) Their difference is $2ph$: $7.3 - 6.5 = 0.8 = 2p(0.2)$, so $p = 2$.

(c) The rate at 3 is $2p(3) + q = 12 + q = 6.9$, so $q = -5.1$.

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