Precalculus Unit 1 worksheet: Graphing polynomial functions (answer key)

ANSWERS AT A GLANCE

1. (a) −3, multiplicity 1, crosses; 0, multiplicity 3, crosses lying flat; 2, multiplicity 1, crosses (b) degree 5; as $x \to -\infty$, $g(x) \to \infty$, and as $x \to \infty$, $g(x) \to -\infty$; y-intercept $(0, 0)$ (c) positive for $x < -3$, negative for $-3 < x < 0$, positive for $0 < x < 2$, negative for $x > 2$; $g(-2) = -8$, $g(-1) = -\frac{3}{2}$, $g(1) = 1$ (d) two turning points: a valley in $-3 < x < 0$ and a peak in $0 < x < 2$.
Features to grade on the sketch: crossings at $(-3, 0)$ and $(2, 0)$; a flat crossing at the origin (the graph levels for an instant and keeps rising); left end up, right end down; the signs in (c); a valley at least 8 deep between −3 and 0 and a peak at least 1 high between 0 and 2; no other turns.

2. (a) $p(x) = x(x-3)(x^2+5)$ (b) zeros 0 and 3, both simple, the graph crosses at each; as $x \to \pm\infty$, $p(x) \to \infty$; y-intercept $(0, 0)$; positive for $x < 0$, negative for $0 < x < 3$, positive for $x > 3$ (c) $p(-1) = 24$, $p(2) = -18$, $p(4) = 84$ (d) one turn, a valley in $0 < x < 3$; no, the sketch cannot rule out a hidden peak and valley pair (calculus shows there are none).
Features to grade on the sketch: crossings at $(0, 0)$ and $(3, 0)$ only; both ends up; a valley between 0 and 3 at least 18 deep; through $(-1, 24)$, $(2, -18)$ and $(4, 84)$.

3. (a) $p(x) = x(x-1)(x+1)(x-3)(x+3)$ (b) zeros −3, −1, 0, 1, 3, all simple; as $x \to -\infty$, $p(x) \to -\infty$, and as $x \to \infty$, $p(x) \to \infty$; negative for $x < -3$, positive on $-3 < x < -1$, negative on $-1 < x < 0$, positive on $0 < x < 1$, negative on $1 < x < 3$, positive for $x > 3$; $p(-2) = 30$, $p(2) = -30$, $p\left(\frac{1}{2}\right) = \frac{105}{32}$ (c) peaks in $-3 < x < -1$ and $0 < x < 1$, valleys in $-1 < x < 0$ and $1 < x < 3$; four turns is $n - 1$ for degree 5.

4. (a) $p(x) = (x-1)(x-3)(x+2)^2$ (b) −2, multiplicity 2, touches and turns back; 1, multiplicity 1, crosses; 3, multiplicity 1, crosses (c) as $x \to \pm\infty$, $p(x) \to \infty$; y-intercept $(0, 12)$; positive for $x < -2$, positive on $-2 < x < 1$, negative on $1 < x < 3$, positive for $x > 3$; $p(-1) = 8$, $p(2) = -16$ (d) the valley $(-2, 0)$ is exact; the peak in $-2 < x < 1$ and the valley in $1 < x < 3$ are only bracketed.
Features to grade on the sketch: a touch from above at $(-2, 0)$; crossings at $(1, 0)$ and $(3, 0)$; both ends up; through $(0, 12)$, $(-1, 8)$ and $(2, -16)$; three turns (valley on the axis, peak, valley) and no more.

5. (a) $f(-3) = 43$, $f(-2) = -1$, $f(-1) = -3$, $f(0) = 1$, $f(1) = -1$, $f(2) = 3$ (b) between −3 and −2, −1 and 0, 0 and 1, 1 and 2; four sign changes give at least four zeros, and degree 4 allows at most four (c) as $x \to \pm\infty$, $f(x) \to \infty$; from left to right the sign is $+, -, +, -, +$; three turns: a valley between the first two zeros, a peak between the middle two, a valley between the last two; three is $n - 1$ for degree 4.

6. (2). (1) touches at 1 and crosses at −2, and gives $y = 1$ at $x = 0$; (3) crosses at 1 lying flat, and gives $y = 25$ at $x = 3$ where the graph is near 6; (4) has both ends rising and gives $y = 2$ at $x = 0$.

WORKED SOLUTIONS

1. Let $g(x) = -\frac{1}{4}x^3(x+3)(x-2)$.

(a) List the zeros of g with their multiplicities, and say what the graph does at each one.

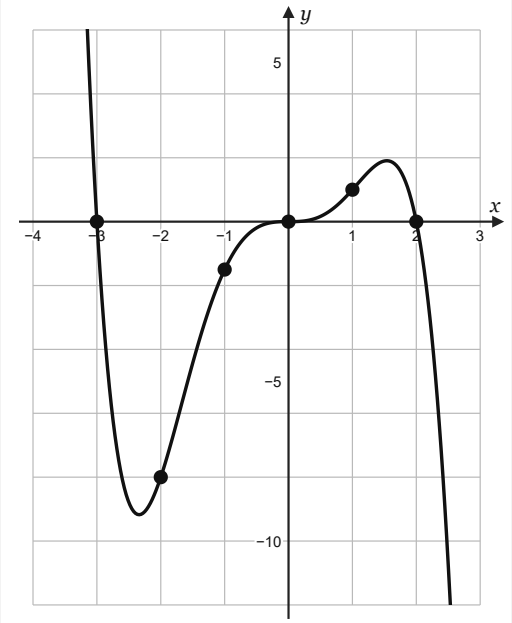
(b) Give the degree and the end behavior of g, and the y-intercept of its graph.

(c) Give the sign of $g(x)$ on each interval between the zeros, and find $g(-2)$, $g(-1)$ and $g(1)$.

(d) Sketch the graph. Degree 5 allows four turning points. How many does your sketch have, and why is the zero at 0 not one of them?

> **ANSWER**
> (a) −3, multiplicity 1, crosses; 0, multiplicity 3, crosses lying flat; 2, multiplicity 1, crosses (b) degree 5; as $x \to -\infty$, $g(x) \to \infty$, and as $x \to \infty$, $g(x) \to -\infty$; y-intercept $(0, 0)$ (c) positive for $x < -3$, negative for $-3 < x < 0$, positive for $0 < x < 2$, negative for $x > 2$; $g(-2) = -8$, $g(-1) = -\frac{3}{2}$, $g(1) = 1$
> (d) two turning points: a valley in $-3 < x < 0$ and a peak in $0 < x < 2$.
> Features to grade on the sketch: crossings at $(-3, 0)$ and $(2, 0)$; a flat crossing at the origin (the graph levels for an instant and keeps rising); left end up, right end down; the signs in (c); a valley at least 8 deep between −3 and 0 and a peak at least 1 high between 0 and 2; no other turns.

Free sample from math.wiki for classroom use. Do not post. math.wiki



SOLUTION

(a) The factor x^3 gives the zero 0 with multiplicity 3: odd, so the graph crosses, and 3 or more, so it lies flat as it crosses. The zeros −3 and 2 are simple: the graph crosses at a slant.

(b) The leading term is $-\frac{1}{4} \cdot x^3 \cdot x \cdot x = -\frac{1}{4}x^5$. Odd degree, negative leading coefficient: as $x \to -\infty$, $g(x) \to \infty$, and as $x \to \infty$, $g(x) \to -\infty$. The y-intercept is $g(0) = 0$, the origin, which is also a zero.

(c) Every zero has odd multiplicity, so the sign flips at each one. For $x > 2$ the three factors x^3, $x + 3$, $x - 2$ are all positive and the $-\frac{1}{4}$ makes g negative. Moving left: positive on $0 < x < 2$, negative on $-3 < x < 0$, positive for $x < -3$. The left end rising agrees.

$g(-2) = -\frac{1}{4}(-8)(1)(-4) = -8$, $g(-1) = -\frac{1}{4}(-1)(2)(-3) = -\frac{3}{2}$, $g(1) = -\frac{1}{4}(1)(4)(-1) = 1$.

(d) From the top left the graph comes down and crosses at −3, dips to a valley in $-3 < x < 0$ (at least 8 deep, since $g(-2) = -8$), and comes up to the origin. Near 0, $g(x) \approx -\frac{1}{4}(3)(-2)x^3 = \frac{3}{2}x^3$, so it passes through the origin lying flat, still going up. It rises to a peak in $0 < x < 2$ (at least 1 high, since $g(1) = 1$), comes down through 2 and falls off to the bottom right.

That is two turns, a valley and a peak. The origin is not a turn: the graph is below the axis just left of 0 and above it just right, so it rises on both sides and only levels for an instant. (Calculus confirms the two turns are the only ones, near $x \approx -2.34$ and $x \approx 1.54$.)

2. Let $p(x) = x^4 - 3x^3 + 5x^2 - 15x$.

(a) Factor $p(x)$ completely over the real numbers.

(b) Give the zeros and what the graph does at each, the end behavior, the y-intercept and the sign of $p(x)$ on each interval between the zeros.

(c) Find $p(-1)$, $p(2)$ and $p(4)$, and sketch the graph.

(d) Degree 4 allows three turning points. How many does your sketch find? Can the sketch alone promise there are no others?

ANSWER

(a) $p(x) = x(x - 3)(x^2 + 5)$ (b) zeros 0 and 3, both simple, the graph crosses at each; as $x \to \pm\infty$, $p(x) \to \infty$; y-intercept (0, 0); positive for $x < 0$, negative for $0 < x < 3$, positive for $x > 3$ (c) $p(-1) =$

Free sample from math.wiki for classroom use. Do not post. math.wiki

24, $p(2) = -18$, $p(4) = 84$ (d) one turn, a valley in $0 < x < 3$; no, the sketch cannot rule out a hidden peak and valley pair (calculus shows there are none).

Features to grade on the sketch: crossings at (0, 0) and (3, 0) only; both ends up; a valley between 0 and 3 at least 18 deep; through (−1, 24), (2, −18) and (4, 84).

SOLUTION

(a) Every term has a factor x: $p(x) = x(x^3 - 3x^2 + 5x - 15)$. Group the cubic in pairs: $x^2(x - 3) + 5(x - 3) = (x^2 + 5)(x - 3)$. So

$$p(x) = x(x - 3)(x^2 + 5).$$

The factor $x^2 + 5$ has discriminant $0 - 20 < 0$, so it does not factor over the real numbers.

(b) The zeros are 0 and 3, both simple: the graph crosses at each. Since $x^2 + 5 \geq 5$, that factor is never 0 and always positive, so it adds no zero and nothing to the sign chart. The leading term is $x \cdot x \cdot x^2 = x^4$: both ends rise. $p(0) = 0$. The sign of p is the sign of $x(x - 3)$: positive for $x < 0$, negative for $0 < x < 3$, positive for $x > 3$.

(c) $p(-1) = (-1)(-4)(6) = 24$, $p(2) = (2)(-1)(9) = -18$, $p(4) = (4)(1)(21) = 84$. The graph comes down from the top left, crosses at the origin, dips to a valley at least 18 deep, comes back up through 3 and rises steeply off to the top right.

(d) The sketch finds one turn, the valley in $0 < x < 3$. One is fewer than $n - 1 = 3$, so a sketch alone cannot rule out a small peak and valley hiding somewhere. Calculus shows there is none here: the graph falls until the valley near $x \approx 1.95$ and rises after it.

3. Let $p(x) = x^5 - 10x^3 + 9x$.

(a) Factor $p(x)$ completely.

(b) Give the zeros, the end behavior and the sign of $p(x)$ on each interval between the zeros. Find $p(-2)$, $p(2)$ and $p\left(\frac{1}{2}\right)$.

(c) List the peaks and valleys of the graph by interval, and explain why it has no others.

ANSWER

(a) $p(x) = x(x - 1)(x + 1)(x - 3)(x + 3)$ (b) zeros −3, −1, 0, 1, 3, all simple; as $x \to -\infty$, $p(x) \to -\infty$, and as $x \to \infty$, $p(x) \to \infty$; negative for $x < -3$, positive on $-3 < x < -1$, negative on $-1 < x < 0$, positive on $0 < x < 1$, negative on $1 < x < 3$, positive for $x > 3$; $p(-2) = 30$, $p(2) = -30$, $p\left(\frac{1}{2}\right) = \frac{105}{32}$

(c) peaks in $-3 < x < -1$ and $0 < x < 1$, valleys in $-1 < x < 0$ and $1 < x < 3$; four turns is $n - 1$ for degree 5.

SOLUTION

(a) Take out x: $p(x) = x(x^4 - 10x^2 + 9)$. The quartic is a quadratic in x^2: $u^2 - 10u + 9 = (u - 1)(u - 9)$. So

$$p(x) = x(x^2 - 1)(x^2 - 9) = x(x - 1)(x + 1)(x - 3)(x + 3).$$

(b) Five simple zeros: −3, −1, 0, 1, 3, each a slanted crossing. The leading term is x^5: the left end falls and the right end rises. For $x > 3$ every factor is positive; each simple zero flips the sign, so from the right the signs are $+, -, +, -, +, -$.

$p(-2) = -32 + 80 - 18 = 30$, $p(2) = 32 - 80 + 18 = -30$ and $p\left(\frac{1}{2}\right) = \frac{1}{32} - \frac{10}{8} + \frac{9}{2} = \frac{1 - 40 + 144}{32} = \frac{105}{32}$.

Free sample from math.wiki for classroom use. Do not post. math.wiki

(c) Between two neighboring zeros the graph leaves the axis and comes back, so it turns at least once: a peak where it is above the axis, a valley where it is below. That gives a peak in $-3 < x < -1$ (at least 30 high), a valley in $-1 < x < 0$, a peak in $0 < x < 1$ (at least $\dfrac{105}{32} \approx 3.3$ high) and a valley in $1 < x < 3$ (at least 30 deep). That is four turns. A polynomial of degree 5 has at most 5 − 1 = 4, so there are no others.

4. The numbers 1 and 3 are zeros of $p(x) = x^4 - 9x^2 - 4x + 12$.

(a) Factor $p(x)$ completely.

(b) Give each zero with its multiplicity, and what the graph does there.

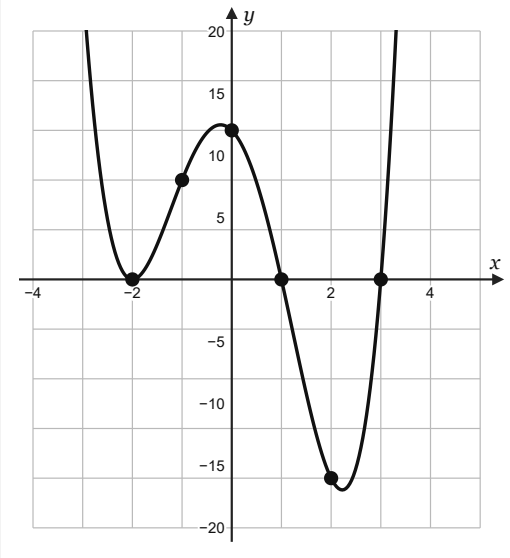
(c) Give the end behavior, the y-intercept and the sign of $p(x)$ on each interval between the zeros. Find $p(-1)$ and $p(2)$.

(d) Sketch the graph. Which of its peaks and valleys does the sketch know exactly, and which only roughly?

ANSWER

(a) $p(x) = (x-1)(x-3)(x+2)^2$ (b) −2, multiplicity 2, touches and turns back; 1, multiplicity 1, crosses; 3, multiplicity 1, crosses (c) as $x \to \pm\infty$, $p(x) \to \infty$; y-intercept (0, 12); positive for $x < -2$, positive on $-2 < x < 1$, negative on $1 < x < 3$, positive for $x > 3$; $p(-1) = 8$, $p(2) = -16$ (d) the valley (−2, 0) is exact; the peak in $-2 < x < 1$ and the valley in $1 < x < 3$ are only bracketed.

Features to grade on the sketch: a touch from above at (−2, 0); crossings at (1, 0) and (3, 0); both ends up; through (0, 12), (−1, 8) and (2, −16); three turns (valley on the axis, peak, valley) and no more.



SOLUTION

(a) Divide by $x - 1$ synthetically, with 0 for the missing x^3 term: the coefficients 1, 0, −9, −4, 12 give 1, 1, −8, −12 with remainder 0, so $p(x) = (x-1)(x^3 + x^2 - 8x - 12)$. Since 3 is also a zero of p and 3 − 1 ≠ 0, it is a zero of the quotient: dividing 1, 1, −8, −12 by 3 gives 1, 4, 4 with remainder 0. Then $x^2 + 4x + 4 = (x+2)^2$, and

$$p(x) = (x-1)(x-3)(x+2)^2.$$

Check. $(x^2 - 4x + 3)(x^2 + 4x + 4) = x^4 - 9x^2 - 4x + 12$.

(b) The zero −2 has multiplicity 2: the graph touches the axis there and turns back. 1 and 3 are simple: it crosses.

(c) The leading term is $x \cdot x \cdot x^2 = x^4$: both ends rise. $p(0) = (-1)(-3)(4) = 12$. For $x > 3$ every factor is positive. The sign flips at 3 and at 1, and does not flip at the double zero −2: positive for $x > 3$, negative on

Free sample from math.wiki for classroom use. Do not post. math.wiki

$1 < x < 3$, positive on $-2 < x < 1$, positive for $x < -2$. $p(-1) = (-2)(-4)(1) = 8$ and $p(2) = (1)(-1)(16) = -16$.

(d) From the top left the graph comes down to touch the axis at $(-2, 0)$ from above, rises to a peak in $-2 < x < 1$ (at least 12 high, since $p(0) = 12$), comes down through 1 to a valley in $1 < x < 3$ (at least 16 deep), and rises through 3. Valley, peak, valley: three turns, $n - 1$ for degree 4.

The valley at $(-2, 0)$ is exact: it is a touching zero. The peak and the other valley are only bracketed by the test values; where each one sits and how far it reaches needs calculus.

5. The polynomial $f(x) = x^4 - 4x^2 + x + 1$ does not factor over the integers, so its zeros cannot be read off.

(a) Make a table of $f(x)$ for $x = -3, -2, -1, 0, 1, 2$.

(b) Between which pairs of consecutive integers does f have a zero? Explain why f has exactly four real zeros.

(c) Give the end behavior and the sign of $f(x)$ between its zeros. How many turning points does the graph have, in which intervals, and why can there be no others?

ANSWER

(a) $f(-3) = 43$, $f(-2) = -1$, $f(-1) = -3$, $f(0) = 1$, $f(1) = -1$, $f(2) = 3$ (b) between −3 and −2, −1 and 0, 0 and 1, 1 and 2; four sign changes give at least four zeros, and degree 4 allows at most four (c) as $x \to \pm\infty$, $f(x) \to \infty$; from left to right the sign is $+, -, +, -, +$; three turns: a valley between the first two zeros, a peak between the middle two, a valley between the last two; three is $n - 1$ for degree 4.

SOLUTION

(a) $f(-3) = 81 - 36 - 3 + 1 = 43$, $f(-2) = 16 - 16 - 2 + 1 = -1$, $f(-1) = 1 - 4 - 1 + 1 = -3$, $f(0) = 1$, $f(1) = 1 - 4 + 1 + 1 = -1$, $f(2) = 16 - 16 + 2 + 1 = 3$.

(b) A polynomial graph has no breaks, so between two inputs where f has opposite signs the graph must cross the axis. The signs change between −3 and −2, between −1 and 0, between 0 and 1, and between 1 and 2: at least one zero in each, so at least four real zeros. A polynomial of degree 4 has at most 4 real zeros, so there are exactly four, one in each interval, and each is simple.

(c) The leading term x^4 makes both ends rise. Each simple zero flips the sign, and the far left is positive, so the signs are $+, -, +, -, +$. Between each pair of neighboring zeros the graph turns: a valley between the zeros in $(-3, -2)$ and $(-1, 0)$ (at least 3 deep, since $f(-1) = -3$), a peak between the zeros in $(-1, 0)$ and $(0, 1)$ (at least 1 high, since $f(0) = 1$), and a valley between the zeros in $(0, 1)$ and $(1, 2)$ (at least 1 deep). Three turns is $n - 1$ for degree 4, so there are no others.

6. The graph below is a polynomial whose x-intercepts are −2, 1 and 4. It passes through the marked points $(0, -2)$ and $(2, 4)$, and the whole graph is in view apart from its two ends.

(1) $y = -\dfrac{1}{8}(x + 2)(x - 1)^2(x - 4)$

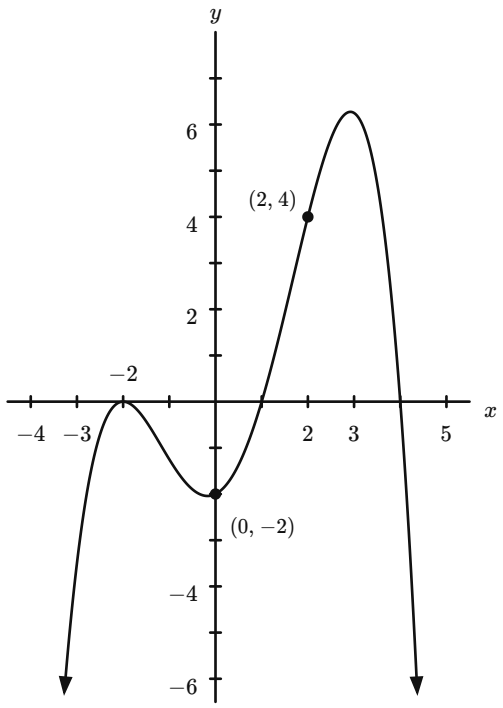
(2) $y = \dfrac{1}{8}(x + 2)^2(x - 1)(4 - x)$

(3) $y = -\dfrac{1}{8}(x + 2)^2(x - 1)^3(x - 4)$

(4) $y = \dfrac{1}{8}(x + 2)^2(x - 1)(x - 4)$

Which rule is the graph? For each of the other three, name a feature of the graph it gets wrong, and show the work.

Free sample from math.wiki for classroom use. Do not post. math.wiki



ANSWER

(2). (1) touches at 1 and crosses at −2, and gives $y = 1$ at $x = 0$; (3) crosses at 1 lying flat, and gives $y = 25$ at $x = 3$ where the graph is near 6; (4) has both ends rising and gives $y = 2$ at $x = 0$.

SOLUTION

Read the graph first. Both ends fall: even degree, negative leading coefficient. At −2 the graph touches and turns back (even multiplicity); at 1 and 4 it crosses at a slant (multiplicity 1). The peak right of 1 is a little above 6.

(2) The factor $4 - x$ has leading term $-x$, so the leading term is $\frac{1}{8} \cdot x^2 \cdot x \cdot (-x) = -\frac{1}{8}x^4$: both ends fall. The square $(x + 2)^2$ gives the touch at −2, and 1 and 4 are simple. At $x = 0$: $\frac{1}{8}(4)(-1)(4) = -2$, and at $x = 2$: $\frac{1}{8}(16)(1)(2) = 4$. Every feature fits: it is the graph.

(1) The square is on $x - 1$, so this graph would touch at 1 and cross at −2, the other way around. Also at $x = 0$ it gives $-\frac{1}{8}(2)(1)(-4) = 1$, not −2.

(3) Its ends fall (leading term $-\frac{1}{8}x^6$), it touches at −2, and it passes through both marked points: $-\frac{1}{8}(4)(-1)(-4) = -2$ and $-\frac{1}{8}(16)(1)(-2) = 4$. But 1 has multiplicity 3, so the graph would lie flat as it crosses there, and the graph shown crosses steeply. The heights disagree too: at $x = 3$ it gives $-\frac{1}{8}(25)(8)(-1) = 25$, while the graph is about 6 there.

(4) Its leading term is $\frac{1}{8}x^4$, so both ends would rise. At $x = 0$ it gives $\frac{1}{8}(4)(-1)(-4) = 2$, not −2.

Free sample from math.wiki for classroom use. Do not post. math.wiki