Precalculus Unit 1 worksheet: Graphing rational functions (answer key)

ANSWERS AT A GLANCE

1. (a) $-$ on $(-\infty, -3)$, $+$ on $(-3, 0)$, $-$ on $(0, 1)$ and $(1, 4)$, $+$ on $(4, \infty)$; no hole (b) $+$ on $(-\infty, -1)$ and $(-1, 2)$, $-$ on $(2, \infty)$; no hole (c) hole at $\left(-2, -\frac{3}{2}\right)$, which does not change the sign; $-$ on $(-\infty, -2)$ and $(-2, 0)$, $+$ on $(0, 2)$, $-$ on $(2, \infty)$

2. (a) $f(x) = \dfrac{(x-3)(x+3)}{x(x-2)(x+2)}$; domain every real number except -2, 0, 2; no holes. At $x = -2$: up on the left, down on the right. At $x = 0$: down on the left, up on the right. At $x = 2$: up on the left, down on the right. Horizontal asymptote $y = 0$. Zeros $x = -3$ and $x = 3$ (crossings); no y-intercept. (b) $-$ on $(-\infty, -3)$, $+$ on $(-3, -2)$, $-$ on $(-2, 0)$, $+$ on $(0, 2)$, $-$ on $(2, 3)$, $+$ on $(3, \infty)$. (c) The graph is unchanged by a half turn about the origin.

3. (a) $x = -1$: $g(x) \to -\infty$ as $x \to -1^-$ and $g(x) \to \infty$ as $x \to -1^+$. (b) $g(x) = x - 3 + \dfrac{4}{x+1}$, so the slant line is $y = x - 3$; above it far right, below it far left; it never meets the line. (c) Zero $x = 1$, a touch; y-intercept $(0, 1)$; $-$ on $(-\infty, -1)$, $+$ on $(-1, 1)$ and $(1, \infty)$.

4. (a) $c = -3$; $f(x) = \dfrac{2x^2 - 7x}{x - 3}$ (b) $x = 3$ and $y = 2x - 1$; the other zero is $x = \dfrac{7}{2}$. (c) Below, since $\dfrac{-3}{x-3} < 0$ for $x > 3$.

5. (1) C (2) A (3) B (4) C (5) none

6. (a) $f(x) = \dfrac{-x(x+1)(x-3)}{x(x-1)^2} = \dfrac{-(x+1)(x-3)}{(x-1)^2}$ for $x \neq 0$; hole at $(0, 3)$. No y-intercept, because 0 is not in the domain. (b) $x - 1$ is left twice below, an even power, and $C = 4 > 0$, so up on both sides; $x + 1$ and $x - 3$ each appear once, an odd power, so the graph crosses at each. (c) The asymptote is $y = -1$, and $f(x) + 1 = \dfrac{4}{(x-1)^2} > 0$, so the graph lies above $y = -1$ everywhere and never meets it.

WORKED SOLUTIONS

1. Without sketching, give the sign ($+$ or $-$) of each function on each interval of its domain between its zeros and vertical asymptotes. Name any hole, and say whether it changes the sign.

(a) $\dfrac{(x+3)(x-1)^2}{x(x-4)}$

(b) $\dfrac{2-x}{(x+1)^2}$

(c) $\dfrac{3x^2 + 6x}{4 - x^2}$

ANSWER

(a) $-$ on $(-\infty, -3)$, $+$ on $(-3, 0)$, $-$ on $(0, 1)$ and $(1, 4)$, $+$ on $(4, \infty)$; no hole (b) $+$ on $(-\infty, -1)$ and $(-1, 2)$, $-$ on $(2, \infty)$; no hole (c) hole at $\left(-2, -\frac{3}{2}\right)$, which does not change the sign; $-$ on $(-\infty, -2)$ and $(-2, 0)$, $+$ on $(0, 2)$, $-$ on $(2, \infty)$

SOLUTION

Start at the far right, where the sign is the sign of the ratio of the leading coefficients, and move left: the sign flips at a break whose factor has an odd power and stays at one whose factor has an even power.

(a) Breaks -3 (power 1), 0 (power 1), 1 (power 2), 4 (power 1). Leading coefficients 1 and 1, so $+$ on $(4, \infty)$. Flip at 4: $-$ on $(1, 4)$. Stay at 1: $-$ on $(0, 1)$. Flip at 0: $+$ on $(-3, 0)$. Flip at -3: $-$ on $(-\infty, -3)$. Check at $x = -1$: $\dfrac{(2)(4)}{(-1)(-5)} = \dfrac{8}{5} > 0$.

(b) Breaks -1 (power 2) and 2 (power 1). Leading coefficients -1 (from $-x$) and 1, so $-$ on $(2, \infty)$. Flip at 2: $+$ on $(-1, 2)$. Stay at -1: $+$ on $(-\infty, -1)$. Check at $x = -2$: $\dfrac{4}{1} > 0$.

(c) Factor first: $\dfrac{3x(x+2)}{(2-x)(2+x)} = \dfrac{3x}{2-x}$ for $x \neq -2$. The factor $x + 2$ canceled completely, so $x = -2$ is a hole, at height $\dfrac{3(-2)}{2+2} = -\dfrac{3}{2}$. The breaks of the lowest-terms formula are 0 and 2. Far right $\dfrac{3x}{2-x}$ has leading coefficients 3 and -1, so $-$ on $(2, \infty)$; flip at 2: $+$ on $(0, 2)$; flip at 0: $-$ on $(-\infty, 0)$. The canceled

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factor $\dfrac{x+2}{x+2}$ is 1 on both sides of -2, so the sign is $-$ on both $(-\infty, -2)$ and $(-2, 0)$. Check at $x = -3$: $\dfrac{-9}{5} < 0$.

2. Let $f(x) = \dfrac{x^2 - 9}{x^3 - 4x}$.

(a) Factor, and give the domain, any holes, each vertical asymptote with what the graph does on each side of it, the horizontal asymptote, and the intercepts.

(b) Give the sign of f on each interval.

(c) Show that $f(-x) = -f(x)$, and say what this tells you about the graph. Then sketch it.

ANSWER

(a) $f(x) = \dfrac{(x-3)(x+3)}{x(x-2)(x+2)}$; domain every real number except -2, 0, 2; no holes. At $x = -2$: up on the left, down on the right. At $x = 0$: down on the left, up on the right. At $x = 2$: up on the left, down on the right. Horizontal asymptote $y = 0$. Zeros $x = -3$ and $x = 3$ (crossings); no y-intercept. (b) $-$ on $(-\infty, -3)$, $+$ on $(-3, -2)$, $-$ on $(-2, 0)$, $+$ on $(0, 2)$, $-$ on $(2, 3)$, $+$ on $(3, \infty)$. (c) The graph is unchanged by a half turn about the origin.

SOLUTION

(a) The numerator is a difference of squares and the denominator has the common factor x: $x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$. So

$$f(x) = \frac{(x-3)(x+3)}{x(x-2)(x+2)}.$$

No factor is common, so there is no hole, and the domain is every real number except -2, 0 and 2. Each is a vertical asymptote with an odd power, so the graph goes off in opposite directions on its two sides. Put the value into every other factor:

At $x = -2$: $C = \dfrac{(-5)(1)}{(-2)(-4)} = -\dfrac{5}{8} < 0$, so $f(x) \to \infty$ as $x \to -2^-$ and $f(x) \to -\infty$ as $x \to -2^+$.

At $x = 0$: $C = \dfrac{(-3)(3)}{(-2)(2)} = \dfrac{9}{4} > 0$, so $f(x) \to -\infty$ as $x \to 0^-$ and $f(x) \to \infty$ as $x \to 0^+$.

At $x = 2$: $C = \dfrac{(-1)(5)}{(2)(4)} = -\dfrac{5}{8} < 0$, so $f(x) \to \infty$ as $x \to 2^-$ and $f(x) \to -\infty$ as $x \to 2^+$.

The numerator has degree 2 and the denominator degree 3, so the horizontal asymptote is $y = 0$. The zeros are -3 and 3, each to the first power, so the graph crosses there. There is no y-intercept: 0 is not in the domain.

(b) The breaks -3, -2, 0, 2, 3 are all odd powers, and the far right is $+$ (leading coefficients 1 and 1), so the signs alternate from the right: $+$ on $(3, \infty)$, $-$ on $(2, 3)$, $+$ on $(0, 2)$, $-$ on $(-2, 0)$, $+$ on $(-3, -2)$, $-$ on $(-\infty, -3)$. Check: $f(1) = \dfrac{-8}{-3} = \dfrac{8}{3} > 0$.

(c) $f(-x) = \dfrac{(-x)^2 - 9}{(-x)^3 - 4(-x)} = \dfrac{x^2 - 9}{-(x^3 - 4x)} = -f(x)$. So for every point (x, y) on the graph, $(-x, -y)$ is on it too: the graph is symmetric about the origin, and the left half is the right half turned half a turn. The signs in (b) agree: they are reversed in mirror-image intervals.

Grade the sketch on: dashed asymptotes $x = -2$, $x = 0$, $x = 2$ and the axis $y = 0$; four branches. Far left, just under the axis, crossing at $(-3, 0)$ and rising to ∞ along $x = -2$. On $(-2, 0)$ a branch below the axis, down along both $x = -2$ and $x = 0$ (through $\left(-1, -\dfrac{8}{3}\right)$). On $(0, 2)$ a branch above the axis, up along

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both $x = 0$ and $x = 2$ (through $\left(1, \frac{8}{3}\right)$). On $(2, \infty)$ a branch up from $-\infty$ along $x = 2$, crossing at (3, 0), then just above the axis and flattening toward it ($f(4) = \frac{7}{48}$). No point on the y-axis.

3. Let $g(x) = \dfrac{x^2 - 2x + 1}{x + 1}$.

(a) Find the vertical asymptote and what the graph does on each side of it.

(b) Use long division to find the line the graph follows far out. Far to the right, is the graph above or below that line? Far to the left? Does the graph ever meet the line?

(c) Find the intercepts and the sign on each interval, saying whether the graph crosses or touches the axis. Then sketch the graph.

ANSWER

(a) $x = -1$: $g(x) \to -\infty$ as $x \to -1^-$ and $g(x) \to \infty$ as $x \to -1^+$. (b) $g(x) = x - 3 + \dfrac{4}{x+1}$, so the slant line is $y = x - 3$; above it far right, below it far left; it never meets the line. (c) Zero $x = 1$, a touch; y-intercept (0, 1); $-$ on $(-\infty, -1)$, $+$ on (−1, 1) and $(1, \infty)$.

SOLUTION

(a) The numerator is $(x-1)^2$, and $x + 1$ is not one of its factors, so nothing cancels: $g(x) = \dfrac{(x-1)^2}{x+1}$. The one vertical asymptote is $x = -1$, an odd power, with $C = (-1-1)^2 = 4 > 0$. As $x \to -1^-$, $x + 1$ is a small negative number, so $g(x) \to -\infty$; as $x \to -1^+$, $g(x) \to \infty$.

(b) The numerator's degree is higher by exactly one. Divide: $x^2 - 2x + 1 = (x+1)(x-3) + 4$, so

$$g(x) = x - 3 + \frac{4}{x+1}.$$

Far out $\dfrac{4}{x+1}$ shrinks to 0, so the graph follows $y = x - 3$. Far right the fraction is a small positive number, so the graph runs just above the line; far left it is a small negative number, so the graph runs just below. The gap $g(x) - (x - 3) = \dfrac{4}{x+1}$ is never 0, so the graph never meets the line.

(c) The zero is $x = 1$, where $x - 1$ appears twice: the graph touches the axis at (1, 0) and turns back. $g(0) = \dfrac{1}{1} = 1$. The breaks are −1 (odd) and 1 (even). Test values: $g(-2) = \dfrac{9}{-1} = -9$, $g(0) = 1$, $g(3) = \dfrac{4}{4} = 1$. So $-$ on $(-\infty, -1)$ and $+$ on (−1, 1) and $(1, \infty)$.

Grade the sketch on: dashed lines $x = -1$ and $y = x - 3$; a left branch, entirely below the axis, coming up from the lower left just under the slant line and falling to $-\infty$ along $x = -1$ (through points such as (−3, −8) and (−2, −9)); a right branch coming down from ∞ along $x = -1$, through (0, 1), touching the axis at (1, 0), rising through (3, 1) and running just above $y = x - 3$ to the upper right; neither branch meets the slant line.

4. A rational function has the form $f(x) = 2x - 1 + \dfrac{c}{x-3}$ for a nonzero constant c, and its graph passes through the origin.

(a) Find c, and write f as a single fraction.

(b) Give the vertical asymptote and the slant asymptote, and find the other zero of f.

(c) Far to the right, is the graph above or below its slant asymptote? Explain.

ANSWER

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(a) $c = -3$; $f(x) = \dfrac{2x^2 - 7x}{x - 3}$ (b) $x = 3$ and $y = 2x - 1$; the other zero is $x = \dfrac{7}{2}$. (c) Below, since $\dfrac{-3}{x - 3} < 0$ for $x > 3$.

SOLUTION

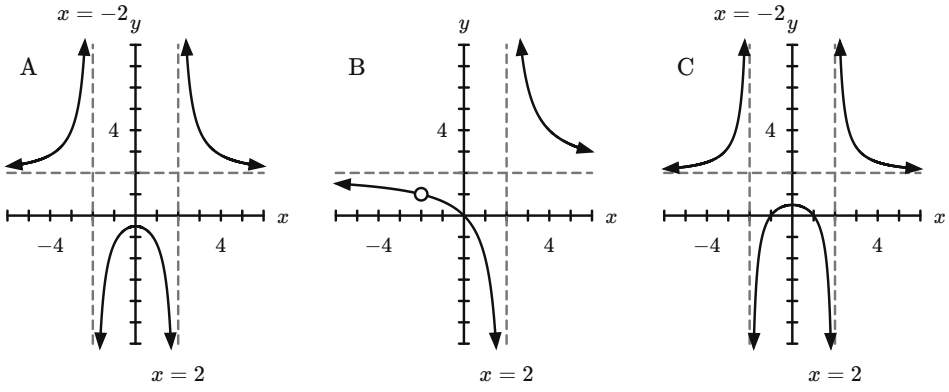
(a) $f(0) = -1 + \dfrac{c}{-3} = 0$ gives $c = -3$. Over the common denominator,

$$f(x) = \frac{(2x - 1)(x - 3) - 3}{x - 3} = \frac{2x^2 - 7x + 3 - 3}{x - 3} = \frac{2x^2 - 7x}{x - 3}.$$

(b) The denominator is zero only at 3, and the numerator $x(2x - 7)$ is $3 \neq 0$ there, so $x = 3$ is the vertical asymptote. Far out $\dfrac{-3}{x - 3}$ shrinks to 0, so the graph follows the slant line $y = 2x - 1$. The zeros are the zeros of $x(2x - 7)$: 0 and $\dfrac{7}{2}$.

(c) For $x > 3$ the gap $f(x) - (2x - 1) = \dfrac{-3}{x - 3}$ is negative, so the graph runs just below the line far right (and just above it far left, where the gap is positive).

5. Each of the graphs A, B and C below is the graph of one of these rules. Asymptotes are dashed, open circles are holes, and tick marks are 1 unit apart.



(1) $\dfrac{2x^2 - 2}{x^2 - 4}$ (2) $\dfrac{2x^2 + 2}{x^2 - 4}$ (3) $\dfrac{2x^2 + 4x}{x^2 - 4}$ (4) $\dfrac{2 - 2x^2}{4 - x^2}$ (5) $\dfrac{2x^2 - 4x}{x^2 - 4}$

One graph fits two of the rules, and one rule fits none of the graphs. Match each rule to its graph, or say that it has none. For each, name the features that decide it.

ANSWER

(1) C (2) A (3) B (4) C (5) none

SOLUTION

Every rule has numerator and denominator of degree 2 with leading coefficients whose ratio is 2 (in (4), $\dfrac{-2}{-1} = 2$), and every graph has horizontal asymptote $y = 2$. So that feature decides nothing; the factors do.

(1) $\dfrac{2(x - 1)(x + 1)}{(x - 2)(x + 2)}$: nothing cancels, vertical asymptotes $x = \pm 2$, zeros $x = \pm 1$, y-intercept $\dfrac{-2}{-4} = \dfrac{1}{2}$. Only C crosses the axis, twice, between its asymptotes: C.

(2) $\dfrac{2(x^2 + 1)}{(x - 2)(x + 2)}$: the numerator is never zero, so there are no zeros; vertical asymptotes $x = \pm 2$; y-intercept $-\dfrac{1}{2}$. A has two vertical asymptotes and never meets the horizontal axis: A.

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(3) $\dfrac{2x(x+2)}{(x-2)(x+2)} = \dfrac{2x}{x-2}$ for $x \neq -2$: a hole at $\left(-2, \dfrac{-4}{-4}\right) = (-2, 1)$, one vertical asymptote $x = 2$, and a zero at the origin. B has one vertical asymptote at 2, passes through the origin and has an open circle at $(-2, 1)$: B.

(4) Multiply top and bottom by -1: $\dfrac{2 - 2x^2}{4 - x^2} = \dfrac{2x^2 - 2}{x^2 - 4}$, and both denominators are zero at exactly ± 2, so (4) is the same function as (1), with the same domain: C.

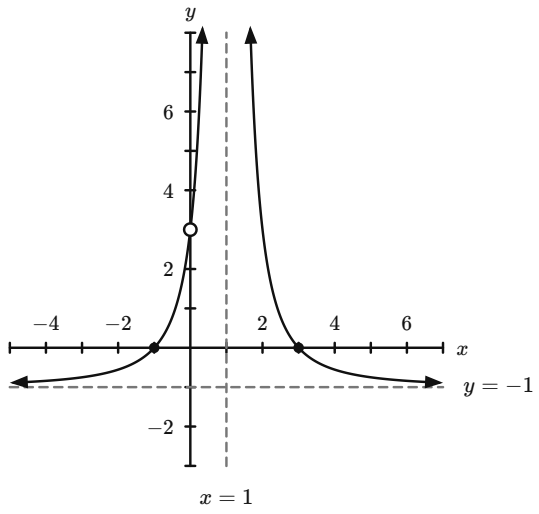
(5) $\dfrac{2x(x-2)}{(x-2)(x+2)} = \dfrac{2x}{x+2}$ for $x \neq 2$: a hole at $(2, 1)$ and one vertical asymptote $x = -2$. No graph has its only vertical asymptote at $x = -2$ (B's is at $x = 2$, and its hole is at $x = -2$), so (5) has no graph.

6. The graph of $f(x) = \dfrac{-x^3 + 2x^2 + 3x}{x^3 - 2x^2 + x}$ is drawn below, with its asymptotes dashed and tick marks 1 unit apart.

(a) Factor, and find the hole. Explain why the graph has no y-intercept, even though it reaches the y-axis.

(b) Use the factors to explain why the graph goes up on both sides of $x = 1$, and why it crosses the axis at $x = -1$ and at $x = 3$.

(c) Show that the graph never meets its horizontal asymptote, and say whether it lies above or below it.



ANSWER

(a) $f(x) = \dfrac{-x(x+1)(x-3)}{x(x-1)^2} = \dfrac{-(x+1)(x-3)}{(x-1)^2}$ for $x \neq 0$; hole at $(0, 3)$. No y-intercept, because 0 is not in the domain. (b) $x - 1$ is left twice below, an even power, and $C = 4 > 0$, so up on both sides; $x + 1$ and $x - 3$ each appear once, an odd power, so the graph crosses at each. (c) The asymptote is $y = -1$, and $f(x) + 1 = \dfrac{4}{(x-1)^2} > 0$, so the graph lies above $y = -1$ everywhere and never meets it.

SOLUTION

(a) Top: $-x(x^2 - 2x - 3) = -x(x+1)(x-3)$. Bottom: $x(x^2 - 2x + 1) = x(x-1)^2$. The factor x cancels completely, so $x = 0$ is a hole, at the height of the canceled formula there: $\dfrac{-(1)(-3)}{(-1)^2} = 3$. The hole is $(0, 3)$. The y-intercept would be $f(0)$, but the original denominator is 0 at $x = 0$, so $f(0)$ does not exist: the graph comes up to $(0, 3)$ from both sides, but that point is missing.

(b) After canceling, $x - 1$ is left twice in the denominator, so $x = 1$ is a vertical asymptote with an even power, and the graph goes the same way on both sides. Put $x = 1$ into the other factors: $C = -(2)(-2) = 4 > 0$, and $\dfrac{4}{(x-1)^2}$ is large and positive on both sides, so the graph goes up on both sides. The zeros are -1 and 3, each from a factor to the first power, an odd power, so the sign changes at each and the graph crosses: $f(-2) = -\dfrac{5}{9}$, $f(2) = 3$, $f(4) = -\dfrac{5}{9}$.

(c) Both degrees are 3, with leading coefficients -1 and 1, so the horizontal asymptote is $y = -1$. The gap from it is

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$$f(x)-(-1)=\frac{-(x+1)(x-3)+(x-1)^2}{(x-1)^2}=\frac{-x^2+2x+3+x^2-2x+1}{(x-1)^2}=\frac{4}{(x-1)^2}.$$

This is never 0, so $f(x)=-1$ has no solution and the graph never meets the asymptote. It is always positive, so both branches lie above $y=-1$.

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