Precalculus Unit 1 worksheet: Polynomial and rational inequalities with sign charts (answer key)

ANSWERS AT A GLANCE

1. (a) $(-5, 0)$ (b) $[-5, 0] \cup \{3\}$ (c) $(-\infty, -5) \cup (0, 3) \cup (3, \infty)$ (d) $(-\infty, -5] \cup [0, \infty)$
2. (a) $(-\infty, -4) \cup (0, 1)$ (b) $\{-2\} \cup [2, \infty)$ (c) $(-\infty, -3) \cup (1, 2)$
3. (a) $[-2, 2)$ (b) $(0, 1] \cup (2, \infty)$ (c) $(-\infty, -2] \cup (-1, 2) \cup (2, \infty)$
4. $\{-6\} \cup [2, 7]$
5. (a) $(-3, 0) \cup (3, \infty)$. (b) It wrongly includes $(-\infty, -3)$ and misses $(-3, 0)$. (c) Dividing by x keeps the direction only when $x > 0$; for $x < 0$ it should reverse.
6. $x \geq 2$: at least 2 liters of tint, the interval $[2, \infty)$.

WORKED SOLUTIONS

1. Let $f(x) = -x(x-3)^2(x+5)$. Make one sign chart for f, then use it to solve all four inequalities. Write each answer in interval notation.

(a) $f(x) > 0$ (b) $f(x) \geq 0$ (c) $f(x) < 0$ (d) $f(x) \leq 0$

ANSWER

(a) $(-5, 0)$ (b) $[-5, 0] \cup \{3\}$ (c) $(-\infty, -5) \cup (0, 3) \cup (3, \infty)$ (d) $(-\infty, -5] \cup [0, \infty)$

SOLUTION

The zeros are -5, 0 and 3. The factors x and $x+5$ have exponent 1; $(x-3)^2$ has exponent 2, so the sign does not change at 3. Far to the right f behaves like $-x^4$, so it is negative there.

	$-x$	$(x-3)^2$	$x+5$	$f(x)$
$(-\infty, -5)$	$+$	$+$	$-$	$-$
$(-5, 0)$	$+$	$+$	$+$	$+$
$(0, 3)$	$-$	$+$	$+$	$-$
$(3, \infty)$	$-$	$+$	$+$	$-$

Test numbers agree: $f(-1) = 64$ and $f(1) = -24$.

(a) Positive only on $(-5, 0)$.

(b) Add the zeros -5, 0 and 3. The zero at 3 sits between two negative stretches, so it is a single point: $[-5, 0] \cup \{3\}$.

(c) Negative on three stretches, with 3 left out because $f(3) = 0$: $(-\infty, -5) \cup (0, 3) \cup (3, \infty)$.

(d) Add the zeros. Now 3 joins $(0, 3)$ and $(3, \infty)$: $(-\infty, -5] \cup [0, \infty)$.

2. Move every term to one side and factor completely, then solve with a sign chart. In (c), find a whole-number zero first.

(a) $2x^3 + 6x^2 < 8x$ (b) $x^3 + 2x^2 - 4x - 8 \geq 0$ (c) $x^3 - 7x + 6 < 0$

ANSWER

(a) $(-\infty, -4) \cup (0, 1)$ (b) $\{-2\} \cup [2, \infty)$ (c) $(-\infty, -3) \cup (1, 2)$

SOLUTION

(a) Subtract $8x$: $2x^3 + 6x^2 - 8x < 0$. Factor out $2x$ (do not divide by x, whose sign changes at 0): $2x(x^2 + 3x - 4) = 2x(x+4)(x-1)$. The critical numbers -4, 0, 1 all have exponent 1 and the leading coefficient is positive, so from the right the signs are $+, -, +, -$. Negative on $(-\infty, -4)$ and $(0, 1)$, zeros left out: $(-\infty, -4) \cup (0, 1)$. Check: $x = \frac{1}{2}$ gives $\frac{1}{4} + \frac{3}{2} = \frac{7}{4} < 4$.

(b) Group: $x^2(x+2) - 4(x+2) = (x+2)(x^2-4) = (x+2)^2(x-2)$. The sign is $+$ on $(2, \infty)$, flips at 2 to $-$ on $(-2, 2)$, and does not flip at -2 (even exponent): $-$ on $(-\infty, -2)$. The product is never positive left

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of 2, but it is 0 at −2, so −2 is a solution on its own: $\{-2\} \cup [2, \infty)$.

(c) Try $x = 1$: 1 − 7 + 6 = 0, so $x - 1$ is a factor. Synthetic division by 1 on 1, 0, −7, 6 gives 1, 1, −6 with remainder 0, so $x^3 - 7x + 6 = (x - 1)(x^2 + x - 6) = (x - 1)(x + 3)(x - 2)$. Three simple zeros −3, 1, 2 and a positive leading coefficient give +, −, +, − from the right. Negative on $(-\infty, -3)$ and (1, 2): $(-\infty, -3) \cup (1, 2)$. Check: $x = \frac{3}{2}$ gives $\frac{27}{8} - \frac{21}{2} + 6 = -\frac{9}{8} < 0$.

3. Solve each inequality. Write it as one fraction compared with 0 first, and say which critical numbers make the expression undefined.

(a) $\dfrac{x^2}{x - 2} \le x + 1$ (b) $\dfrac{1}{x} + \dfrac{1}{x - 2} \ge 0$ (c) $\dfrac{x^2 - 4}{x^2 - x - 2} \ge 0$

ANSWER

(a) $[-2, 2)$ (b) $(0, 1] \cup (2, \infty)$ (c) $(-\infty, -2] \cup (-1, 2) \cup (2, \infty)$

SOLUTION

(a) Subtract $x + 1$ over the denominator $x - 2$:

$$\frac{x^2 - (x + 1)(x - 2)}{x - 2} = \frac{x^2 - x^2 + x + 2}{x - 2} = \frac{x + 2}{x - 2} \le 0.$$

Zero at −2, undefined at 2. The fraction is negative between them. Include −2, never 2: $[-2, 2)$. Check: $x = 0$ gives 0 ≤ 1, and $x = -3$ gives $-\frac{9}{5} \le -2$, false.

(b) Common denominator $x(x - 2)$:

$$\frac{(x - 2) + x}{x(x - 2)} = \frac{2(x - 1)}{x(x - 2)} \ge 0.$$

Zero at 1; undefined at 0 and 2. From the right: + on $(2, \infty)$, − on (1, 2), + on (0, 1), − on $(-\infty, 0)$. Include 1 only: $(0, 1] \cup (2, \infty)$.

(c) Factor: $\dfrac{(x - 2)(x + 2)}{(x - 2)(x + 1)}$. The denominator is zero at 2 and −1, so both are out, even though the factor $x - 2$ cancels. For $x \ne 2$ the fraction equals $\dfrac{x + 2}{x + 1}$, which is + on $(-1, \infty)$, − on (−2, −1) and + on $(-\infty, -2)$, with a zero at −2. Near 2 it is about $\frac{4}{3}$ on both sides, so the hole does not change the sign; it only removes one point:

$$(-\infty, -2] \cup (-1, 2) \cup (2, \infty).$$

4. Solve $(x + 6)^2(x - 2)(x - 7) \le 0$. Write the answer in interval notation.

ANSWER

$\{-6\} \cup [2, 7]$

SOLUTION

The critical numbers are −6, 2 and 7. Far to the right every factor is positive, so the product is + on $(7, \infty)$. The sign flips at 7 and at 2, which have exponent 1: − on (2, 7) and + on (−6, 2). The factor $(x + 6)^2$ has an even exponent, so the sign does not flip at −6: + on $(-\infty, -6)$.

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The product is negative only on (2, 7). Because the inequality allows 0, all three zeros are in, and −6 stands alone between two positive stretches:

$$\{-6\} \cup [\,2, 7\,].$$

5. Samir solves $x^3 > 9x$ like this: "Divide both sides by x to get $x^2 > 9$, so the answer is $(-\infty, -3) \cup (3, \infty)$."

(a) Solve $x^3 > 9x$ correctly with a sign chart.

(b) Which numbers does Samir's answer wrongly include, and which solutions does it miss?

(c) Explain which step went wrong and why.

ANSWER

(a) $(-3, 0) \cup (3, \infty)$. (b) It wrongly includes $(-\infty, -3)$ and misses (−3, 0). (c) Dividing by x keeps the direction only when $x > 0$; for $x < 0$ it should reverse.

SOLUTION

(a) Subtract $9x$ and factor: $x^3 - 9x = x(x + 3)(x - 3) > 0$. Three simple zeros −3, 0, 3 and a positive leading coefficient give, from the right, $+$ on $(3, \infty)$, $-$ on (0, 3), $+$ on (−3, 0) and $-$ on $(-\infty, -3)$. So the solution set is $(-3, 0) \cup (3, \infty)$.

(b) The two answers agree on $(3, \infty)$. Samir's answer wrongly includes $(-\infty, -3)$: for example $x = -4$ gives −64 > −36, which is false. It misses (−3, 0): for example $x = -1$ gives −1 > −9, which is true.

(c) Dividing both sides by x is correct only where x is positive. For $x < 0$, dividing by x reverses the inequality, so on the negative side the true condition is $x^2 < 9$, which gives (−3, 0). Since x is an expression whose sign changes at 0, the safe route is to subtract and factor instead of dividing.

6. A mixing tank at a paint store holds 16 liters of paint, and 4 liters of it are blue tint. The clerk pours in x liters of pure blue tint, so the fraction of the paint that is blue becomes $\dfrac{4 + x}{16 + x}$. How much tint must be added for blue to make up at least $\dfrac{1}{3}$ of the paint? Solve the inequality with a sign chart.

ANSWER

$x \geq 2$: at least 2 liters of tint, the interval $[\,2, \infty)$.

SOLUTION

Solve $\dfrac{4 + x}{16 + x} \geq \dfrac{1}{3}$ for $x \geq 0$. Subtract the right side and use the common denominator $3(16 + x)$:

$$\frac{3(4 + x) - 1(16 + x)}{3(16 + x)} = \frac{2x - 4}{3(16 + x)} \geq 0.$$

The numerator is zero at $x = 2$ and the denominator at $x = -16$, which is not an amount of tint. For every $x \geq 0$ the denominator is positive, so the fraction has the sign of the numerator: negative on $[0, 2)$ and positive on $(2, \infty)$. At $x = 2$ the fraction is 0, which the inequality allows.

So $x \geq 2$. Check: with 2 liters added, $\dfrac{6}{18} = \dfrac{1}{3}$ exactly.

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