Precalculus Unit 1 worksheet: Slant asymptotes (answer key)

ANSWERS AT A GLANCE

1. (a) $m = \frac{3}{2}$ (b) $m = -\frac{1}{2}$ (c) $m = \frac{2}{3}$ (d) $m = -\frac{1}{4}$
2. (a) $x - 1 + \frac{4}{x-3}$; slant asymptote $y = x - 1$; vertical asymptote $x = 3$.
   (b) $2x - 3 + \frac{7}{3x+2}$; slant asymptote $y = 2x - 3$; vertical asymptote $x = -\frac{2}{3}$.
   (c) $x + 2 - \frac{4x+1}{x^2+3}$; slant asymptote $y = x + 2$; no vertical asymptote.
   (d) $-2x - 1 - \frac{x-4}{x^2-x}$; slant asymptote $y = -2x - 1$; vertical asymptotes $x = 0$ and $x = 1$.
3. (a) Horizontal asymptote $y = \frac{5}{2}$.
   (b) The graph is the line $y = x + 4$ with a hole at $(3, 7)$; no asymptote.
   (c) No line: the graph follows a parabola, $y = x^2 + 5$.
   (d) Slant asymptote $y = 3x - 2$.
4. Slant asymptote $y = 2x + 4$; vertical asymptote $x = \frac{5}{2}$.
5. (a) $y = 0.02n + 5$ (b) $F = 500$ dollars (c) 12 dollars
6. (a) The gap $h(x) - (3x + 1) = 5(0.5)^x$ shrinks toward 0 as $x \to \infty$ and is always positive, so the graph is above the line; at $x = 10$ the gap is $\frac{5}{1024} \approx 0.0049$. (b) At $x = -10$ the gap is $5{,}120$; the gap grows without bound as $x \to -\infty$, so the graph does not approach the line on the left.

WORKED SOLUTIONS

1. Each function has a slant asymptote. Without dividing, give the slope of that asymptote.

(a) $\frac{6x^3 - x + 2}{4x^2 + 1}$ (b) $\frac{-5x^2 + 2x}{10x + 3}$ (c) $\frac{2x^5 + 1}{3x^4 - x^3}$

(d) $\frac{x^2 + 7}{-4x + 1}$

ANSWER

(a) $m = \frac{3}{2}$ (b) $m = -\frac{1}{2}$ (c) $m = \frac{2}{3}$ (d) $m = -\frac{1}{4}$

SOLUTION

The slope is the leading coefficient of the numerator divided by the leading coefficient of the denominator, because the first step of the division divides the two leading terms.

(a) $6x^3 \div 4x^2 = \frac{3}{2}x$, so $m = \frac{3}{2}$.

(b) $-5x^2 \div 10x = -\frac{1}{2}x$, so $m = -\frac{1}{2}$.

(c) $2x^5 \div 3x^4 = \frac{2}{3}x$, so $m = \frac{2}{3}$.

(d) $x^2 \div (-4x) = -\frac{1}{4}x$, so $m = -\frac{1}{4}$. The leading coefficient of $-4x + 1$ is −4, not 1.

2. Divide, and write each function as a quotient plus a remainder over the denominator. Then give the slant asymptote and any vertical asymptotes.

(a) $\frac{x^2 - 4x + 7}{x - 3}$ (b) $\frac{6x^2 - 5x + 1}{3x + 2}$

(c) $\frac{x^3 + 2x^2 - x + 5}{x^2 + 3}$ (d) $\frac{-2x^3 + x^2 + 4}{x^2 - x}$

ANSWER

(a) $x - 1 + \frac{4}{x-3}$; slant asymptote $y = x - 1$; vertical asymptote $x = 3$.

(b) $2x - 3 + \frac{7}{3x+2}$; slant asymptote $y = 2x - 3$; vertical asymptote $x = -\frac{2}{3}$.

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(c) $x + 2 - \dfrac{4x+1}{x^2+3}$; slant asymptote $y = x + 2$; no vertical asymptote.

(d) $-2x - 1 - \dfrac{x-4}{x^2-x}$; slant asymptote $y = -2x - 1$; vertical asymptotes $x = 0$ and $x = 1$.

SOLUTION

(a) $x^2 \div x = x$, and $x(x-3) = x^2 - 3x$ leaves $-x + 7$; then $-x \div x = -1$, and $-1(x-3) = -x + 3$ leaves 4. So the function is $x - 1 + \dfrac{4}{x-3}$ and the slant asymptote is $y = x - 1$. At $x = 3$ the numerator is 9 − 12 + 7 = 4 ≠ 0, so $x = 3$ is a vertical asymptote.

(b) $6x^2 \div 3x = 2x$, and $2x(3x+2) = 6x^2 + 4x$ leaves $-9x + 1$; then $-9x \div 3x = -3$, and $-3(3x+2) = -9x - 6$ leaves 7. So the function is $2x - 3 + \dfrac{7}{3x+2}$ and the slant asymptote is $y = 2x - 3$. The denominator is 0 at $x = -\dfrac{2}{3}$, where the numerator is $\dfrac{8}{3} + \dfrac{10}{3} + 1 = 7 \neq 0$: a vertical asymptote.

(c) $x^3 \div x^2 = x$, and $x(x^2+3) = x^3 + 3x$ leaves $2x^2 - 4x + 5$; then $2x^2 \div x^2 = 2$, and $2(x^2+3) = 2x^2 + 6$ leaves $-4x - 1$. So the function is $x + 2 - \dfrac{4x+1}{x^2+3}$ and the slant asymptote is $y = x + 2$. Since $x^2 + 3 > 0$ for every x, there is no vertical asymptote.

(d) $-2x^3 \div x^2 = -2x$, and $-2x(x^2-x) = -2x^3 + 2x^2$ leaves $-x^2 + 4$; then $-x^2 \div x^2 = -1$, and $-1(x^2-x) = -x^2 + x$ leaves $-x + 4$. So the function is $-2x - 1 - \dfrac{x-4}{x^2-x}$ and the slant asymptote is $y = -2x - 1$. The denominator $x(x-1)$ is 0 at $x = 0$ and $x = 1$, where the numerator is 4 and 3. Both are vertical asymptotes.

3. For each function, say what its graph does far out: approaches a horizontal line, approaches a slant line, follows a curve that is not a line, or is itself a line with a hole. Give the line when there is one.

(a) $\dfrac{5x^2 - 1}{2x^2 + x}$

(b) $\dfrac{x^2 + x - 12}{x - 3}$

(c) $\dfrac{x^4 + x}{x^2 - 5}$

(d) $\dfrac{3x^3 - 2x^2 + 1}{x^2 + 4}$

ANSWER

(a) Horizontal asymptote $y = \dfrac{5}{2}$.

(b) The graph is the line $y = x + 4$ with a hole at (3, 7); no asymptote.

(c) No line: the graph follows a parabola, $y = x^2 + 5$.

(d) Slant asymptote $y = 3x - 2$.

SOLUTION

(a) The degrees are equal, so the asymptote is horizontal, at the ratio of the leading coefficients: $y = \dfrac{5}{2}$.

(b) The numerator factors as $(x+4)(x-3)$, so the division leaves remainder 0: the function equals $x + 4$ for $x \neq 3$. The graph is the line itself, with a hole at $x = 3$, $y = 3 + 4 = 7$. A graph that is the line does not approach it, so this is not a slant asymptote.

(c) The degrees are 4 and 2, two apart. Division gives $x^4 + x = (x^2 - 5)(x^2 + 5) + x + 25$, so the function is $x^2 + 5 + \dfrac{x + 25}{x^2 - 5}$. The quotient has degree 2, so the graph follows the parabola $y = x^2 + 5$ and no line.

(d) The degrees are 3 and 2, one apart. Division: $3x(x^2+4) = 3x^3 + 12x$ leaves $-2x^2 - 12x + 1$; then $-2(x^2+4) = -2x^2 - 8$ leaves $-12x + 9$. The remainder is not 0, so the slant asymptote is $y = 3x - 2$.

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4. The denominator of $f(x) = \dfrac{4x^2 - 2x - 18}{2x - 5}$ is 0 at $x = \dfrac{5}{2}$. Use synthetic division by $\dfrac{5}{2}$ to find the slant asymptote of f and its vertical asymptote. Watch the leading coefficient 2 of the denominator.

ANSWER

Slant asymptote $y = 2x + 4$; vertical asymptote $x = \dfrac{5}{2}$.

SOLUTION

Synthetic division of 4, −2, −18 by $\dfrac{5}{2}$: bring down 4; $\dfrac{5}{2} \cdot 4 = 10$, and −2 + 10 = 8; $\dfrac{5}{2} \cdot 8 = 20$, and −18 + 20 = 2. So

$$4x^2 - 2x - 18 = \left(x - \frac{5}{2}\right)(4x + 8) + 2.$$

Since $2x - 5 = 2\left(x - \dfrac{5}{2}\right)$, move the 2 into the denominator by halving the quotient: the numerator is $(2x - 5)(2x + 4) + 2$, and

$$f(x) = 2x + 4 + \frac{2}{2x - 5}.$$

The slant asymptote is $y = 2x + 4$, not $y = 4x + 8$. The remainder 2 is not 0, so the numerator is not 0 at $x = \dfrac{5}{2}$ and the vertical asymptote is $x = \dfrac{5}{2}$.

5. Diego runs the math club's calendar fundraiser. Printing n calendars costs $C(n) = 0.02n^2 + 5n + F$ dollars: a one-time setup fee of F dollars, 5 dollars of paper and ink per calendar, and a binding charge that grows faster for big orders. The average cost per calendar is $A(n) = \dfrac{C(n)}{n}$.

(a) Find the slant asymptote of A.

(b) For an order of 100 calendars, the print shop's quote shows an average cost 5 dollars above the slant line. Find the setup fee F.

(c) Find the average cost per calendar for that order.

ANSWER

(a) $y = 0.02n + 5$ (b) $F = 500$ dollars (c) 12 dollars

SOLUTION

(a) Divide each term by n: $A(n) = 0.02n + 5 + \dfrac{F}{n}$. The leftover $\dfrac{F}{n}$ shrinks as n grows, so the slant asymptote is $y = 0.02n + 5$.

(b) The gap between the average and the line is $\dfrac{F}{n}$. At $n = 100$ it is 5 dollars: $\dfrac{F}{100} = 5$, so $F = 500$ dollars.

(c) The line gives 0.02(100) + 5 = 2 + 5 = 7 dollars, and the average is 5 dollars above it: 12 dollars. Check: $C(100) \div 100 = 12$.

6. Let $h(x) = 3x + 1 + 5(0.5)^x$. This is not a rational function.

(a) Show that $y = 3x + 1$ is a slant asymptote of h on the right, and say which side of the line the graph is on there. Find the gap at $x = 10$.

(b) Find the gap at $x = -10$, and explain why $y = 3x + 1$ is not an asymptote on the left.

ANSWER

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(a) The gap $h(x) - (3x + 1) = 5(0.5)^x$ shrinks toward 0 as $x \to \infty$ and is always positive, so the graph is above the line; at $x = 10$ the gap is $\dfrac{5}{1024} \approx 0.0049$. (b) At $x = -10$ the gap is 5,120; the gap grows without bound as $x \to -\infty$, so the graph does not approach the line on the left.

SOLUTION

(a) The vertical gap between the graph and the line is

$$h(x) - (3x + 1) = 5(0.5)^x.$$

Each step of 1 to the right halves it, so it shrinks toward 0 as $x \to \infty$: at $x = 10$ it is $\dfrac{5}{2^{10}} = \dfrac{5}{1024} \approx 0.0049$. That is the definition of a slant asymptote on the right. A power of 0.5 is always positive, so the graph is above the line.

(b) $(0.5)^{-10} = 2^{10} = 1024$, so the gap at $x = -10$ is 5 · 1024 = 5,120. Each step to the left doubles the gap, so it grows without bound as $x \to -\infty$ and the graph pulls away from the line. A rational function cannot do this, because one division identity holds for every x and serves both ends; h has no such identity.

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