Cosine and sine as coordinates on the unit circle

A right triangle only holds acute angles. How do you find the sine, cosine and tangent of any angle at all, even an obtuse angle, a negative angle or one past a full turn?

WORDS TO KNOW

Unit circle: the circle of radius 1 centered at the origin. Its points satisfy $x^2 + y^2 = 1$.

Terminal point: the point where the terminal side of an angle in standard position meets the unit circle.

$\cos\theta$**:** the x-coordinate of the terminal point of θ.

$\sin\theta$**:** the y-coordinate of the terminal point of θ.

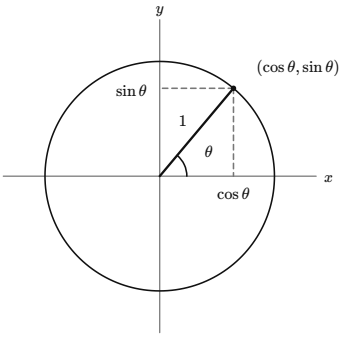
$\tan\theta$**:** the y-coordinate divided by the x-coordinate, the slope of the terminal side.

Undefined: having no value. $\tan\theta$ is undefined when $x = 0$.

KEY IDEAS

1. The terminal point is $(\cos\theta, \sin\theta)$

Every angle in standard position, of any size or sign, has a terminal side, and it meets the unit circle at exactly one point. Its coordinates are the cosine and the sine. For an acute angle these equal the right-triangle ratios, because the hypotenuse is a radius of length 1.



If the terminal point of θ is (x, y), then

$$\cos\theta = x \qquad \sin\theta = y \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

The point is on the unit circle, so $\cos^2\theta + \sin^2\theta = 1$, and both values lie between -1 and 1.

A negative cosine just means the point is left of the y-axis. A negative sine means it is below the x-axis.

2. Any point on the terminal side works

Dividing a point's coordinates by its distance from the origin slides it along the same ray to distance 1, onto the unit circle.

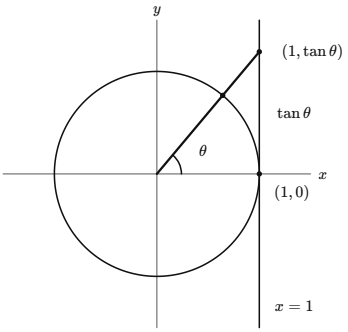
If (x, y) is on the terminal side, not the origin, and $r = \sqrt{x^2 + y^2}$, then

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x}$$

On the circle of radius r centered at the origin, the terminal side meets the circle at $(r\cos\theta,\ r\sin\theta)$.

3. The tangent is a slope

The ratio $\frac{y}{x}$ is rise over run from the origin, the slope of the terminal side. The line through the terminal side meets the line $x = 1$ at the point $(1, \tan\theta)$ when $x \neq 0$.



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WATCH OUT

- **One coordinate does not fix the other's sign.** From $\cos\theta$ alone, $\sin\theta = \pm\sqrt{1-\cos^2\theta}$. Use the quadrant to choose the sign.
- **Divide by r, not by 1.** Unless the point is on the unit circle, its coordinates are not the cosine and sine. They can even be bigger than 1.
- $\tan\theta$ **needs** $x \neq 0$**.** On the y-axis the slope is undefined, so the tangent is too. Cosine and sine are defined for every angle.
- **Match the calculator mode to the angle.** $\cos 2$ means 2 radians. A degree angle always carries its $^\circ$.

WORKED EXAMPLES

Example 1. The terminal side of θ passes through $(-48, 55)$. Find $\cos\theta$, $\sin\theta$ and $\tan\theta$, and name the quadrant.

1. Find the distance from the origin:
$$r = \sqrt{(-48)^2 + 55^2} = \sqrt{2304 + 3025} = \sqrt{5329} = 73$$
2. Divide each coordinate by r: the terminal point is $\left(-\frac{48}{73}, \frac{55}{73}\right)$.
3. The tangent is y over x, and needs no r: $\tan\theta = \frac{55}{-48} = -\frac{55}{48}$.
4. $x < 0$ and $y > 0$, so the point is up and to the left.

Answer: $\cos\theta = -\frac{48}{73}$, $\sin\theta = \frac{55}{73}$, $\tan\theta = -\frac{55}{48}$, Quadrant II. **Check:** $\frac{2304}{5329} + \frac{3025}{5329} = 1$, so the terminal point is on the unit circle.

Example 2. θ lies in Quadrant IV and $\cos\theta = \frac{39}{89}$. Find $\sin\theta$ and $\tan\theta$.

1. The terminal point is on the unit circle, so
$$\sin^2\theta = 1 - \left(\frac{39}{89}\right)^2 = 1 - \frac{1521}{7921} = \frac{6400}{7921}, \qquad \sin\theta = \pm\frac{80}{89}$$
2. Two points of the circle have $x = \frac{39}{89}$, one above the x-axis and one below.
3. Quadrant IV is below the x-axis, so $\sin\theta = -\frac{80}{89}$.
4. Then y over x:
$$\tan\theta = \frac{-80/89}{39/89} = -\frac{80}{39}$$

Answer: $\sin\theta = -\frac{80}{89}$ and $\tan\theta = -\frac{80}{39}$.

Example 3. A rider on a merry-go-round sits 6 m from the center and starts due east. The ride turns 232° counterclockwise. Where is the rider, to the nearest centimeter?

1. Put the center at the origin with east along the positive x-axis. The rider ends on the terminal side of 232°, at distance 6.
2. So the position is $(6\cos 232^\circ,\ 6\sin 232^\circ)$. Use the calculator in degree mode.
$$x = 6\cos 232^\circ \approx 6(-0.61566) \approx -3.69 \qquad y = 6\sin 232^\circ \approx 6(-0.78801) \approx -4.73$$
3. Both coordinates are negative, as they should be: 232° is in Quadrant III.

Answer: about (−3.69, −4.73), so 3.69 m west and 4.73 m south of the center. **Check:** $3.69^2 + 4.73^2 \approx 35.99$, close to $6^2 = 36$.

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