Precalculus Unit 3 worksheet: Trig functions on the unit circle (answer key)

ANSWERS AT A GLANCE

1. (a) $\cos\theta = -\frac{5}{13}, \sin\theta = -\frac{12}{13}, \tan\theta = \frac{12}{5}$ (b) $\cos\theta = \frac{20}{29}, \sin\theta = -\frac{21}{29}, \tan\theta = -\frac{21}{20}$ (c) $\cos\theta = -\frac{15}{17}, \sin\theta = \frac{8}{17}, \tan\theta = -\frac{8}{15}$ (d) $\cos\theta = -\frac{3\sqrt{13}}{13}, \sin\theta = -\frac{2\sqrt{13}}{13}, \tan\theta = \frac{2}{3}$ (e) $\cos\theta = \frac{\sqrt{3}}{2}, \sin\theta = -\frac{1}{2}, \tan\theta = -\frac{\sqrt{3}}{3}$ (f) $\cos\theta = 0, \sin\theta = -1$, $\tan\theta$ is undefined

2. (a) (−4.70, −1.71) (b) (−9.90, 1.41) (c) (2.17, −5.59) (d) (−1.37, 3.76)

3. $\sin\theta = -\frac{\sqrt{15}}{4}, \tan\theta = -\sqrt{15}$

4. $k = 10, \sin\theta = -\frac{12}{13}, \tan\theta = -\frac{12}{5}$

5. (a) $\left(\frac{8}{17}, -\frac{15}{17}\right)$ (b) (16, −30) and $\left(\frac{24}{17}, -\frac{45}{17}\right)$ (c) (24, −45), at distance 51

6. (a) $\cos\theta = \frac{12}{13}, \sin\theta = -\frac{5}{13}, \tan\theta = -\frac{5}{12}$ (b) (36, −15) (c) $\left(1, -\frac{5}{12}\right)$

7. (a) (−6.3, 13.6) (b) (−3.2, 6.8) (c) (−3.9, −14.5)

WORKED SOLUTIONS

1. The terminal side of an angle θ in standard position passes through the given point. Find $\cos\theta$, $\sin\theta$ and $\tan\theta$ exactly, with rational denominators.

(a) $(-5, -12)$ (b) $(20, -21)$ (c) $(-15, 8)$

(d) $(-6, -4)$ (e) $(\sqrt{3}, -1)$ (f) $(0, -3)$

ANSWER

(a) $\cos\theta = -\frac{5}{13}, \sin\theta = -\frac{12}{13}, \tan\theta = \frac{12}{5}$ (b) $\cos\theta = \frac{20}{29}, \sin\theta = -\frac{21}{29}, \tan\theta = -\frac{21}{20}$ (c) $\cos\theta = -\frac{15}{17}, \sin\theta = \frac{8}{17}, \tan\theta = -\frac{8}{15}$ (d) $\cos\theta = -\frac{3\sqrt{13}}{13}, \sin\theta = -\frac{2\sqrt{13}}{13}, \tan\theta = \frac{2}{3}$ (e) $\cos\theta = \frac{\sqrt{3}}{2}, \sin\theta = -\frac{1}{2}, \tan\theta = -\frac{\sqrt{3}}{3}$ (f) $\cos\theta = 0, \sin\theta = -1$, $\tan\theta$ is undefined

SOLUTION

For a point (x, y) on the terminal side, find its distance $r = \sqrt{x^2 + y^2}$ from the origin; then $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ and $\tan\theta = \frac{y}{x}$.

(a) $r = \sqrt{25 + 144} = 13$, so $\cos\theta = -\frac{5}{13}, \sin\theta = -\frac{12}{13}, \tan\theta = \frac{-12}{-5} = \frac{12}{5}$.

(b) $r = \sqrt{400 + 441} = \sqrt{841} = 29$, so $\cos\theta = \frac{20}{29}, \sin\theta = -\frac{21}{29}, \tan\theta = -\frac{21}{20}$.

(c) $r = \sqrt{225 + 64} = \sqrt{289} = 17$, so $\cos\theta = -\frac{15}{17}, \sin\theta = \frac{8}{17}, \tan\theta = -\frac{8}{15}$.

(d) $r = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$, so $\cos\theta = \frac{-6}{2\sqrt{13}} = -\frac{3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}, \sin\theta = \frac{-4}{2\sqrt{13}} = -\frac{2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}, \tan\theta = \frac{-4}{-6} = \frac{2}{3}$.

(e) $r = \sqrt{3 + 1} = 2$, so $\cos\theta = \frac{\sqrt{3}}{2}, \sin\theta = -\frac{1}{2}, \tan\theta = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

(f) $r = 3$, so $\cos\theta = \frac{0}{3} = 0$ and $\sin\theta = \frac{-3}{3} = -1$. The point is on the y-axis, so $x = 0$ and $\tan\theta = \frac{y}{x}$ is undefined.

2. Find, to the nearest hundredth, the point where the terminal side of θ meets the circle of radius r centered at the origin. Use a calculator, in degree mode for angles in degrees and in radian mode for the others.

(a) $r = 5, \theta = 200^\circ$ (b) $r = 10, \theta = 3$

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(c) $r = 6, \theta = -1.2$ (d) $r = 4, \theta = 470^\circ$

ANSWER
(a) (−4.70, −1.71) (b) (−9.90, 1.41) (c) (2.17, −5.59) (d) (−1.37, 3.76)

SOLUTION
The point is $(r\cos\theta, r\sin\theta)$. Round only at the end.
(a) Degree mode: $5\cos 200^\circ \approx 5(-0.93969) \approx -4.70$ and $5\sin 200^\circ \approx 5(-0.34202) \approx -1.71$.
(b) Radian mode: $10\cos 3 \approx 10(-0.98999) \approx -9.90$ and $10\sin 3 \approx 10(0.14112) \approx 1.41$.
(c) Radian mode: $6\cos(-1.2) \approx 6(0.36236) \approx 2.17$ and $6\sin(-1.2) \approx 6(-0.93204) \approx -5.59$.
(d) Degree mode: $4\cos 470^\circ \approx 4(-0.34202) \approx -1.37$ and $4\sin 470^\circ \approx 4(0.93969) \approx 3.76$.

3. The terminal point of an angle θ has x-coordinate $\dfrac{1}{4}$ and lies below the x-axis. Find $\sin\theta$ and $\tan\theta$ exactly.

ANSWER
$\sin\theta = -\dfrac{\sqrt{15}}{4}$, $\tan\theta = -\sqrt{15}$

SOLUTION
The terminal point is on the unit circle, and its x-coordinate is $\cos\theta$. Since $\cos^2\theta + \sin^2\theta = 1$,

$$\sin^2\theta = 1 - \frac{1}{16} = \frac{15}{16}, \qquad \sin\theta = \pm\frac{\sqrt{15}}{4}$$

The point lies below the x-axis, so its y-coordinate is negative: $\sin\theta = -\dfrac{\sqrt{15}}{4}$. Then

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\sqrt{15}$$

4. The terminal side of an angle θ passes through the point $(k, -24)$, and $\cos\theta = \dfrac{5}{13}$. Find k, $\sin\theta$ and $\tan\theta$.

ANSWER
$k = 10$, $\sin\theta = -\dfrac{12}{13}$, $\tan\theta = -\dfrac{12}{5}$

SOLUTION
Let r be the distance from the origin to $(k, -24)$. Then $\sin\theta = \dfrac{-24}{r}$, which has the same sign as −24. Since $\cos^2\theta + \sin^2\theta = 1$,

$$\sin^2\theta = 1 - \frac{25}{169} = \frac{144}{169}, \qquad \sin\theta = -\frac{12}{13}$$

So $r = \dfrac{-24}{\sin\theta} = 26$, and $k = r\cos\theta = 10$. Then

$$\tan\theta = \frac{-24}{10} = -\frac{12}{5}$$

Check: $\sqrt{100 + 576} = 26$.

5. The terminal side of an angle θ crosses the circle of radius 17 centered at the origin at the point (8, −15).
(a) Find the terminal point of θ.

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(b) Find where the terminal side crosses the circle of radius 34 and the circle of radius 3, both centered at the origin.

(c) The terminal side also passes through a point with y-coordinate -45. Find that point and its distance from the origin.

ANSWER

(a) $\left(\frac{8}{17}, -\frac{15}{17}\right)$ (b) $(16, -30)$ and $\left(\frac{24}{17}, -\frac{45}{17}\right)$ (c) $(24, -45)$, at distance 51

SOLUTION

(a) The point $(8, -15)$ is at distance 17. Dividing both coordinates by 17 slides it along the same ray to distance 1: the terminal point is $\left(\frac{8}{17}, -\frac{15}{17}\right)$, so $\cos\theta = \frac{8}{17}$ and $\sin\theta = -\frac{15}{17}$.

(b) On the circle of radius r the terminal side passes through $(r\cos\theta, r\sin\theta)$. For $r = 34$: $\left(34 \cdot \frac{8}{17}, 34 \cdot \left(-\frac{15}{17}\right)\right) = (16, -30)$. For $r = 3$: $\left(\frac{24}{17}, -\frac{45}{17}\right)$.

(c) A point of the terminal side at distance r has $y = r\sin\theta$, so

$$-45 = r \cdot \left(-\frac{15}{17}\right), \qquad r = 51$$

Its x-coordinate is $51 \cdot \frac{8}{17} = 24$, so the point is $(24, -45)$, at distance 51. Check: $\sqrt{24^2 + 45^2} = \sqrt{2601} = 51$.

6. The terminal side of an angle θ lies along the line $5x + 12y = 0$ and is below the x-axis.

(a) Find $\cos\theta$, $\sin\theta$ and $\tan\theta$.

(b) Where does the terminal side meet the circle of radius 39 centered at the origin?

(c) Where does the line containing the terminal side meet the vertical line $x = 1$?

ANSWER

(a) $\cos\theta = \frac{12}{13}$, $\sin\theta = -\frac{5}{13}$, $\tan\theta = -\frac{5}{12}$ (b) $(36, -15)$ (c) $\left(1, -\frac{5}{12}\right)$

SOLUTION

(a) The line is $y = -\frac{5}{12}x$. Below the x-axis $y < 0$, so $x > 0$: take $x = 12$, giving $y = -5$, and $(12, -5)$ is on the terminal side. Its distance from the origin is $r = \sqrt{144 + 25} = 13$, so

$$\cos\theta = \frac{12}{13}, \qquad \sin\theta = -\frac{5}{13}, \qquad \tan\theta = \frac{-5}{12} = -\frac{5}{12}$$

the slope of the line.

(b) $(39\cos\theta, 39\sin\theta) = \left(39 \cdot \frac{12}{13}, 39 \cdot \left(-\frac{5}{13}\right)\right) = (36, -15)$.

(c) A line through the origin with slope $\tan\theta$ has height $\tan\theta$ at $x = 1$, so it meets $x = 1$ at $\left(1, -\frac{5}{12}\right)$. Here the terminal side itself reaches $x = 1$, since it points to the right.

7. The glass plate in a microwave oven turns on a round turntable. Seen from above, put the center of the plate at the origin, with lengths in centimeters. The plate has radius 15 cm and turns counterclockwise.

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(a) A mug sits at the edge of the plate at (15, 0), and the plate turns 115°. Find the mug's new position to the nearest tenth of a centimeter (use a calculator).

(b) A small cup sits on the same radius as the mug, halfway between the center and the edge, and turns with the plate. Where is the cup after the same turn?

(c) The plate then turns a further 140°. Where is the mug now, to the nearest tenth of a centimeter?

ANSWER

(a) (−6.3, 13.6) (b) (−3.2, 6.8) (c) (−3.9, −14.5)

SOLUTION

(a) The mug stays 15 cm from the center on the terminal side of 115°, so it is at $(15\cos 115^\circ, 15\sin 115^\circ)$. In degree mode,

$$15\cos 115^\circ \approx 15(-0.42262) \approx -6.3 \qquad 15\sin 115^\circ \approx 15(0.90631) \approx 13.6$$

The mug is at about (−6.3, 13.6).

(b) The cup is on the same terminal side, 7.5 cm from the center: $(7.5\cos 115^\circ, 7.5\sin 115^\circ) \approx (-3.2, 6.8)$, half of each coordinate of the mug's exact position.

(c) From (15, 0) the mug has now turned 115° + 140° = 255° in all, so it is at

$$(15\cos 255^\circ, 15\sin 255^\circ) \approx (15(-0.25882), 15(-0.96593)) \approx (-3.9, -14.5)$$

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