Verify an identity by working on one side

How do you prove that two trig expressions are equal at every angle where both exist, and how do you show that a claimed identity is false?

WORDS TO KNOW

Identity: an equation that is true at every input where both sides are defined.

Verify: prove that the identity holds on its whole common domain.

Conjugate: the same two terms with the sign between them flipped, such as $1 + \sin x$ for $1 - \sin x$.

k: any integer.

Common domain: the inputs where both sides are defined. The identity claims nothing outside it.

Chain: a list of equal expressions from one side to the other, each step an identity you already trust.

Counterexample: one input where both sides are defined and give different values.

KEY IDEAS

1. Work from one side until it becomes the other

Start with one side, usually the more complicated one. Rewrite it one step at a time until it is exactly the other side. Each step swaps an expression for an equal one, so the first expression equals the last. You may also rewrite each side into the same expression C: every step is an equality, so the second half read backward joins the two into one chain.

$$\text{left side} = A_1 = A_2 = \cdots = \text{right side}$$

Find the common domain first. $\tan x$ and $\sec x$ need $\cos x \neq 0$, $\cot x$ and $\csc x$ need $\sin x \neq 0$, and no denominator may be 0. Cancel a factor only where it is not 0.

2. The moves that usually work

When unsure, rewrite everything in sine and cosine. Keep the other side in view: if it is one fraction, combine; if it has only $\sin x$, turn each $\cos^2 x$ into $1 - \sin^2 x$.

Sine and cosine:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Trade a square: $\sin^2 x + \cos^2 x = 1$, $1 + \tan^2 x = \sec^2 x$, $1 + \cot^2 x = \csc^2 x$.

Conjugate pairs: $(1 - \sin x)(1 + \sin x) = \cos^2 x$ and $(1 - \cos x)(1 + \cos x) = \sin^2 x$. Multiplying top and bottom by a nonzero conjugate multiplies by 1.

3. One counterexample disproves

An identity claims equality at every input of the common domain. So one input where both sides are defined and disagree shows it is false, and many matching values still prove nothing. Test at $\frac{\pi}{6}$ or $\frac{\pi}{3}$, where $\sin x$ and $\cos x$ differ and neither is 0. At 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$ or π, different expressions often give the same number.

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WATCH OUT

- **Never square or cross-multiply both sides of the claim.** Squaring loses the sign. $\cos x = \sqrt{1-\sin^2 x}$ squares to a true identity, yet at $x = \pi$ the sides are -1 and 1. The root of a square is an absolute value: $\sqrt{\cos^2 x} = |\cos x|$.
- **An input where one side has no value is not a counterexample.** It is outside the common domain, and the identity says nothing there.
- **Do not cancel a factor that can be 0.** Check that the common domain keeps it nonzero.
- **A failed chain proves nothing.** Only a counterexample shows a claim is false.

WORKED EXAMPLES

Example 1. Verify $\dfrac{\sec x + \tan x}{\cot x + \cos x} = \tan x \sec x$ and state the common domain.

1. Domain: $\sec x$ and $\tan x$ need $\cos x \neq 0$, and $\cot x$ needs $\sin x \neq 0$. The bottom is 0 only when $\cos x = 0$ or $\sin x = -1$ (step 2), and $\sin x = -1$ forces $\cos x = 0$ anyway. So $x \neq \frac{k\pi}{2}$.
2. Rewrite the top and the bottom in sine and cosine, and combine each:
$$\sec x + \tan x = \frac{1+\sin x}{\cos x}$$
$$\cot x + \cos x = \frac{\cos x + \sin x \cos x}{\sin x} = \frac{\cos x\,(1+\sin x)}{\sin x}$$
3. Dividing by the bottom means multiplying by its flip. Cancel $1+\sin x$, which is not 0 on the domain:
$$\frac{1+\sin x}{\cos x} \cdot \frac{\sin x}{\cos x\,(1+\sin x)} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x$$

Answer: an identity, for $x \neq \frac{k\pi}{2}$. **Check:** at $\frac{\pi}{6}$ the top is $\sqrt{3}$ and the bottom is $\frac{3\sqrt{3}}{2}$, so the left side is $\frac{2}{3}$. The right side is $\frac{\sqrt{3}}{3} \cdot \frac{2\sqrt{3}}{3} = \frac{2}{3}$.

Example 2. Verify $\dfrac{\cos x}{1-\sin x} - \dfrac{\cos x}{1+\sin x} = 2\tan x$ and state the common domain.

1. Domain: the bottoms need $\sin x \neq \pm 1$, and $\tan x$ needs $\cos x \neq 0$. These fail at the same inputs. The common domain is $x \neq \frac{\pi}{2} + k\pi$.
2. The common denominator is a conjugate pair, $(1-\sin x)(1+\sin x) = 1 - \sin^2 x = \cos^2 x$:
$$\frac{\cos x\,(1+\sin x) - \cos x\,(1-\sin x)}{(1-\sin x)(1+\sin x)} = \frac{2\sin x \cos x}{\cos^2 x}$$
3. Cancel $\cos x$, which is not 0 on the domain. What is left is $2 \cdot \frac{\sin x}{\cos x} = 2\tan x$.

Answer: an identity, for $x \neq \frac{\pi}{2} + k\pi$. **Check:** at $\frac{\pi}{6}$ the left side is $\sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$, and $2\tan\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$.

Example 3. Both sides of $\sec x = \cos x$ are equal at $x = 0$ and at $x = \pi$. Is it an identity?

1. The common domain is $x \neq \frac{\pi}{2} + k\pi$. At 0 and π, $\cos x$ is 1 or -1, and each of these numbers is its own reciprocal: a poor test.
2. Try $\frac{\pi}{3}$, where both sides are defined: $\sec\frac{\pi}{3} = 2$, but $\cos\frac{\pi}{3} = \frac{1}{2}$.
3. Since $2 \neq \frac{1}{2}$, the two sides disagree at an input of the common domain.

Answer: not an identity. $x = \frac{\pi}{3}$ is a counterexample.

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