Precalculus Unit 5 worksheet: Verifying identities (answer key)

ANSWERS AT A GLANCE

1. Each chain runs from the left side to the right side. Common domains: (a) $x \neq \frac{k\pi}{2}$ (b) $x \neq k\pi$ (c) $x \neq \frac{k\pi}{2}$ (d) $x \neq \frac{\pi}{2} + k\pi$
2. (a) $x \neq \frac{k\pi}{2}$ (b) $x \neq k\pi$; each chain runs from the left side to the right side.
3. (a) Identity, for $x \neq \frac{k\pi}{2}$. (b) Not an identity: at $x = \frac{\pi}{3}$ the left side is $-\frac{1}{2}$ and the right side is 1.
4. (a) The top is $(\csc x - 1)^2$ and the bottom is $\csc^2 x - 1 = (\csc x - 1)(\csc x + 1)$; cancel $\csc x - 1$. Common domain: $x \neq \frac{k\pi}{2}$. (b) At $\frac{\pi}{2}$, $\cot x = 0$, so the left side has no value; the input is outside the common domain, and the identity says nothing there.
5. The denominator becomes $(\sin x + \cos x - 1)(\sin x + 1) = \cos x\,(\sin x - \cos x + 1)$, and the factor $\sin x - \cos x + 1$ cancels. Common domain: $x \neq \frac{\pi}{2} + k\pi$ and $x \neq 2k\pi$.
6. (a) For example $x = \pi$: the left side is -1 and the right side is 1. (b) She worked on both sides of an unproved claim; squaring cannot be run backward, since $a^2 = b^2$ allows $a = -b$. (c) $\sqrt{1 + 2\sin x \cos x} = \sqrt{(\sin x + \cos x)^2} = |\sin x + \cos x|$, so the claim holds exactly where $\sin x + \cos x \geq 0$.

WORKED SOLUTIONS

1. Verify each identity by working on one side only, naming the identity used at each step. State the common domain, with k standing for any integer.

(a) $\sec x \csc x \cos^2 x = \cot x$ (b) $(1 - \sin^2 x)(1 + \cot^2 x) = \cot^2 x$ (c) $\sec^2 x \cot^2 x - \cot^2 x = 1$

(d) $\dfrac{\tan^2 x}{\sec x} = \sec x - \cos x$

ANSWER

Each chain runs from the left side to the right side. Common domains: (a) $x \neq \frac{k\pi}{2}$ (b) $x \neq k\pi$ (c) $x \neq \frac{k\pi}{2}$ (d) $x \neq \frac{\pi}{2} + k\pi$

SOLUTION

(a) **Domain.** $\sec x$ needs $\cos x \neq 0$ and $\csc x$, $\cot x$ need $\sin x \neq 0$: $x \neq \frac{k\pi}{2}$. Rewrite in sine and cosine (reciprocal identities), then cancel the nonzero $\cos x$:

$$\sec x \csc x \cos^2 x = \frac{1}{\cos x} \cdot \frac{1}{\sin x} \cdot \cos^2 x = \frac{\cos x}{\sin x} = \cot x$$

The last step is the quotient identity.

(b) **Domain.** Only $\cot x$ can fail: $x \neq k\pi$. Two Pythagorean trades, $1 - \sin^2 x = \cos^2 x$ and $1 + \cot^2 x = \csc^2 x$, then the reciprocal and quotient identities:

$$(1 - \sin^2 x)(1 + \cot^2 x) = \cos^2 x \csc^2 x = \frac{\cos^2 x}{\sin^2 x} = \cot^2 x$$

(c) **Domain.** $\sec x$ needs $\cos x \neq 0$ and $\cot x$ needs $\sin x \neq 0$: $x \neq \frac{k\pi}{2}$. Factor out $\cot^2 x$, trade $\sec^2 x - 1 = \tan^2 x$ (Pythagorean), and use $\cot x \tan x = 1$ (reciprocal):

$$\sec^2 x \cot^2 x - \cot^2 x = \cot^2 x\,(\sec^2 x - 1) = \cot^2 x \tan^2 x = 1$$

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(d) **Domain.** $\tan x$ and $\sec x$ need $\cos x \neq 0$, and $\sec x$ is never 0: $x \neq \dfrac{\pi}{2} + k\pi$. Trade $\tan^2 x = \sec^2 x - 1$ (Pythagorean), split the fraction, and use the reciprocal identity:

$$\frac{\tan^2 x}{\sec x} = \frac{\sec^2 x - 1}{\sec x} = \sec x - \frac{1}{\sec x} = \sec x - \cos x$$

2. Verify each identity and state its common domain. Combine fractions, or multiply by a conjugate, where that helps.

(a) $\dfrac{1}{\sec x - \cos x} = \cot x \csc x$ (b) $\dfrac{\sin x}{\csc x + \cot x} = 1 - \cos x$

ANSWER

(a) $x \neq \dfrac{k\pi}{2}$ (b) $x \neq k\pi$; each chain runs from the left side to the right side.

SOLUTION

(a) **Domain.** $\sec x$ needs $\cos x \neq 0$, and the denominator $\sec x - \cos x = \dfrac{\sin^2 x}{\cos x}$ (the chain below) is 0 when $\sin x = 0$. The right side needs $\sin x \neq 0$ too. So $x \neq \dfrac{k\pi}{2}$. Combine the denominator over $\cos x$ and trade $1 - \cos^2 x = \sin^2 x$ (Pythagorean):

$$\frac{1}{\sec x - \cos x} = \frac{1}{\frac{1-\cos^2 x}{\cos x}} = \frac{1}{\frac{\sin^2 x}{\cos x}} = \frac{\cos x}{\sin^2 x} = \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = \cot x \csc x$$

(quotient and reciprocal identities in the last step).

(b) **Domain.** $\csc x$ and $\cot x$ need $\sin x \neq 0$. The denominator $\csc x + \cot x = \dfrac{1 + \cos x}{\sin x}$ is 0 only when $\cos x = -1$, already excluded. So $x \neq k\pi$. The pair $\csc x \pm \cot x$ has product $\csc^2 x - \cot^2 x = 1$, so multiply top and bottom by the conjugate $\csc x - \cot x = \dfrac{1 - \cos x}{\sin x}$, which is 0 only when $\cos x = 1$, also excluded:

$$\frac{\sin x}{\csc x + \cot x} = \frac{\sin x \left(\csc x - \cot x\right)}{\csc^2 x - \cot^2 x} = \frac{\sin x \csc x - \sin x \cot x}{1} = 1 - \cos x$$

using $\sin x \csc x = 1$ (reciprocal) and $\sin x \cot x = \cos x$ (quotient).

3. Decide whether each equation is an identity. Verify it, with its common domain, or give a counterexample.

(a) $\dfrac{\cot x - \tan x}{\sin x \cos x} = \csc^2 x - \sec^2 x$ (b) $\cos^4 x - \sin^4 x = 1$

ANSWER

(a) Identity, for $x \neq \dfrac{k\pi}{2}$. (b) Not an identity: at $x = \dfrac{\pi}{3}$ the left side is $-\dfrac{1}{2}$ and the right side is 1.

SOLUTION

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(a) **Test.** At $x = \dfrac{\pi}{3}$: $\cot x - \tan x = \dfrac{\sqrt{3}}{3} - \sqrt{3} = -\dfrac{2\sqrt{3}}{3}$ and $\sin x \cos x = \dfrac{\sqrt{3}}{4}$, so the left side is $-\dfrac{2\sqrt{3}}{3} \cdot \dfrac{4}{\sqrt{3}} = -\dfrac{8}{3}$; the right side is $\dfrac{4}{3} - 4 = -\dfrac{8}{3}$. It passes, so prove it. **Domain.** $\cot x$ needs $\sin x \neq 0$, $\tan x$ needs $\cos x \neq 0$, and so does the divisor: $x \neq \dfrac{k\pi}{2}$. **Chain.** Rewrite in sine and cosine, combine, then split:

$$\frac{\cot x - \tan x}{\sin x \cos x} = \frac{\frac{\cos^2 x - \sin^2 x}{\sin x \cos x}}{\sin x \cos x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} = \csc^2 x - \sec^2 x$$

(b) **Test.** Both sides exist for every x. At $x = 0$ both sides are 1, which proves nothing: there $\sin x = 0$, and many expressions collapse to the same number. Try $x = \dfrac{\pi}{3}$: $\cos^4 x = \left(\dfrac{1}{2}\right)^4 = \dfrac{1}{16}$ and $\sin^4 x = \left(\dfrac{\sqrt{3}}{2}\right)^4 = \dfrac{9}{16}$, so the left side is $-\dfrac{8}{16} = -\dfrac{1}{2}$, while the right side is 1. Both are defined and differ, so it is not an identity. (Factoring the difference of squares shows what the left side really is: $(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) = \cos^2 x - \sin^2 x$.)

4. (a) Verify $\dfrac{\csc^2 x - 2\csc x + 1}{\cot^2 x} = \dfrac{\csc x - 1}{\csc x + 1}$ and state the common domain.

(b) At $x = \dfrac{\pi}{2}$ the right side has the value 0. Explain why this does not contradict the identity, and why it is not a counterexample either.

ANSWER

(a) The top is $(\csc x - 1)^2$ and the bottom is $\csc^2 x - 1 = (\csc x - 1)(\csc x + 1)$; cancel $\csc x - 1$. Common domain: $x \neq \dfrac{k\pi}{2}$. (b) At $\dfrac{\pi}{2}$, $\cot x = 0$, so the left side has no value; the input is outside the common domain, and the identity says nothing there.

SOLUTION

(a) **Domain.** $\csc x$ and $\cot x$ need $\sin x \neq 0$, and the denominator needs $\cot x \neq 0$, that is $\cos x \neq 0$. The right side needs $\csc x \neq -1$, that is $\sin x \neq -1$, which is already excluded. The common domain is $x \neq \dfrac{k\pi}{2}$.

Factor the top as a perfect square in $\csc x$, trade $\cot^2 x = \csc^2 x - 1$ (Pythagorean), and factor the difference of squares:

$$\frac{\csc^2 x - 2\csc x + 1}{\cot^2 x} = \frac{(\csc x - 1)^2}{\csc^2 x - 1} = \frac{(\csc x - 1)^2}{(\csc x - 1)(\csc x + 1)} = \frac{\csc x - 1}{\csc x + 1}$$

The factor $\csc x - 1$ is 0 only when $\sin x = 1$, which is outside the domain, so the cancellation is allowed.

(b) At $x = \dfrac{\pi}{2}$, $\csc x = 1$, so the right side is $\dfrac{0}{2} = 0$. But $\cot \dfrac{\pi}{2} = 0$, so the left side has denominator 0 and no value. A counterexample needs an input where both sides are defined and disagree; here one side is undefined, so the input is outside the common domain and the identity makes no claim about it.

5. Verify $\dfrac{\sin x - \cos x + 1}{\sin x + \cos x - 1} = \dfrac{\sin x + 1}{\cos x}$ and state the common domain. (Hint: multiply the top and bottom of the left side by $\sin x + 1$, and expand only the denominator.)

ANSWER

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The denominator becomes $(\sin x + \cos x - 1)(\sin x + 1) = \cos x\,(\sin x - \cos x + 1)$, and the factor $\sin x - \cos x + 1$ cancels. Common domain: $x \neq \dfrac{\pi}{2} + k\pi$ and $x \neq 2k\pi$.

SOLUTION

Domain. The left side needs $\sin x + \cos x \neq 1$. On one turn $\sin x + \cos x = 1$ at $x = 0$ and $x = \dfrac{\pi}{2}$ (squaring, $2\sin x \cos x = 0$, so one of them is 0 and the other is 1). The right side needs $\cos x \neq 0$. The common domain is $x \neq \dfrac{\pi}{2} + k\pi$ and $x \neq 2k\pi$.

Two factors that must not be zero. $\sin x + 1 = 0$ only at $x = \dfrac{3\pi}{2} + 2k\pi$, where $\cos x = 0$: excluded. $\sin x - \cos x + 1 = 0$ means $\cos x - \sin x = 1$, which on one turn happens at $x = 0$ and $x = \dfrac{3\pi}{2}$: both excluded. So both steps below are valid on the whole domain.

Chain. Multiply top and bottom by $\sin x + 1$ (multiplying by 1), expand the denominator, and trade $\sin^2 x - 1 = -\cos^2 x$ (Pythagorean):

$$\begin{aligned}\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} &= \frac{(\sin x - \cos x + 1)(\sin x + 1)}{(\sin x + \cos x - 1)(\sin x + 1)} \\ &= \frac{(\sin x - \cos x + 1)(\sin x + 1)}{\sin^2 x - 1 + \sin x \cos x + \cos x} \\ &= \frac{(\sin x - \cos x + 1)(\sin x + 1)}{-\cos^2 x + \sin x \cos x + \cos x} \\ &= \frac{(\sin x - \cos x + 1)(\sin x + 1)}{\cos x\,(\sin x - \cos x + 1)} = \frac{\sin x + 1}{\cos x}\end{aligned}$$

The chain runs from the left side to the right side, so the identity holds on the common domain.

6. Elena claims that $\sin x + \cos x = \sqrt{1 + 2\sin x \cos x}$ for every x. Her argument: "Square both sides. The left becomes $\sin^2 x + 2\sin x \cos x + \cos^2 x = 1 + 2\sin x \cos x$, which is the right side squared. True, so the claim holds."

(a) Give a counterexample to her claim.

(b) Explain what is wrong with her argument.

(c) Write a correct chain that starts from the right side, and use it to state a condition on $\sin x + \cos x$ that tells exactly when her claim is true.

ANSWER

(a) For example $x = \pi$: the left side is -1 and the right side is 1. (b) She worked on both sides of an unproved claim; squaring cannot be run backward, since $a^2 = b^2$ allows $a = -b$. (c) $\sqrt{1 + 2\sin x \cos x} = \sqrt{(\sin x + \cos x)^2} = |\sin x + \cos x|$, so the claim holds exactly where $\sin x + \cos x \geq 0$.

SOLUTION

(a) Both sides exist for every x, since $1 + 2\sin x \cos x = (\sin x + \cos x)^2 \geq 0$. At $x = \pi$: $\sin \pi + \cos \pi = 0 + (-1) = -1$, while $\sqrt{1 + 2 \cdot 0 \cdot (-1)} = \sqrt{1} = 1$. Since $-1 \neq 1$, the claim is false.

(b) She started from the claim, applied an operation to both sides, and reached a true statement. That runs in the wrong direction: true statements can follow from false ones (squaring $-1 = 1$ gives $1 = 1$). A proof needs a chain from known facts to the claim, and squaring cannot be undone, because $a^2 = b^2$ says only that $a = b$ or $a = -b$. Her work shows only that the two sides are equal or opposite.

(c) Start from the right side, write 1 as $\sin^2 x + \cos^2 x$ (Pythagorean), recognize a perfect square, and use $\sqrt{u^2} = |u|$:

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$$\sqrt{1+2\sin x\cos x}=\sqrt{\sin^2 x+2\sin x\cos x+\cos^2 x}=\sqrt{(\sin x+\cos x)^2}=|\sin x+\cos x|$$

So her right side equals $|\sin x+\cos x|$ for every x. It equals $\sin x+\cos x$ exactly when $\sin x+\cos x\geq 0$, and her claim fails wherever $\sin x+\cos x<0$.

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