The law of cosines

The law of sines needs a side and the angle across from it. How do you solve a triangle when you know two sides and the angle between them, or all three sides?

WORDS TO KNOW

SAS: two sides and the angle between them, the **included** angle.

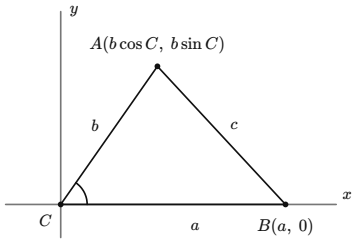
SSS: all three sides, and no angle.

Side names: side a is across from angle A, side b across from B, side c across from C.

Inverse cosine $\cos^{-1}$**:** gives the one angle from 0° to 180° with a given cosine.

KEY IDEAS

1. The law, from coordinates



Put C at the origin and $B = (a, 0)$ on the positive x-axis. Side CA has length b at the angle C, so $A = (b\cos C,\ b\sin C)$. The distance formula gives c^2:

$$c^2 = (b\cos C - a)^2 + (b\sin C)^2$$

Multiply out and use $\cos^2 C + \sin^2 C = 1$.

The law of cosines. In any triangle ABC,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad b^2 = a^2 + c^2 - 2ac\cos B$$

The angle is always the one **between** the two sides on the right, across from the side on the left.

This works for an obtuse C too: then $\cos C < 0$ and A sits left of the y-axis, but its coordinates are still $(b\cos C,\ b\sin C)$.

2. The Pythagorean theorem is inside it

When $C = 90°$, $\cos C = 0$ and the law becomes $c^2 = a^2 + b^2$. The term $-2ab\cos C$ corrects for an angle that is not right.

Let c be the longest side. Compare c^2 with $a^2 + b^2$:

less: the triangle is **acute**. Equal: **right**. Greater: **obtuse**.

3. An angle from three sides

Solve the law for the cosine. Each cosine from −1 to 1 belongs to exactly one angle from 0° to 180°, so the inverse cosine never misleads. A negative cosine means an obtuse angle.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

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4. Which law first, and the safe angle

Given	Name	Start with
two angles and a side	AAS, ASA	the law of sines
two sides and the angle between them	SAS	the law of cosines
three sides	SSS	the law of cosines, largest angle first

After an SAS triangle gives its third side, find the **smaller** remaining angle (across from the shorter given side) by the law of sines. It must be acute, so the inverse sine returns it correctly. Then use the angle sum.

WATCH OUT

- **Use the angle between the two sides.** In $c^2 = a^2 + b^2 - 2ab\cos C$, the angle C sits where a and b meet. Any other angle gives a wrong length.
- **An inverse sine can hide an obtuse angle.** It always returns an angle under 90°. Use it only on an angle you know is acute, or use the inverse cosine.
- **A negative cosine makes the term add.** $-2ab\cos C$ is positive when C is obtuse, so the third side is longer than a right angle would make it. Watch the sign of the cosine.
- **Three sides must make a triangle.** Each side must be less than the sum of the other two. If not, the formula gives a cosine outside −1 to 1, or exactly 1 or −1 (a flat figure), and no angle of a triangle has it.

WORKED EXAMPLES

Example 1. In triangle ABC, $b = 5$, $c = 12$ and $A = 33^\circ$. Solve the triangle. Round sides to two decimal places and angles to hundredths of a degree.

1. A is between b and c (SAS), so use the form with a^2:
$$a^2 = 5^2 + 12^2 - 2(5)(12)\cos 33^\circ = 169 - 120\cos 33^\circ \approx 68.36, \qquad a \approx 8.27$$
2. The shorter given side is 5, so B (across from it) is the smaller remaining angle and is acute. Law of sines:
$$\sin B = \frac{5\sin 33^\circ}{a} \approx 0.3294, \qquad B \approx 19.23^\circ$$
3. Angle sum: $C = 180^\circ - 33^\circ - 19.23^\circ = 127.77^\circ$.
4. Check C by cosines, which cannot mislead:
$$\cos C = \frac{5^2 + a^2 - 12^2}{2(5)(a)} \approx -0.6125, \qquad C \approx 127.77^\circ$$

Answer: $a \approx 8.27$, $B \approx 19.23^\circ$, $C \approx 127.77^\circ$. An inverse sine at C would have given 52.23°, the wrong angle.

Example 2. Solve the triangle with $a = 10$, $b = 13$ and $c = 17$. Give exact cosines and angles to hundredths of a degree.

1. The longest side is $c = 17$, so find C first:
$$\cos C = \frac{10^2 + 13^2 - 17^2}{2(10)(13)} = \frac{-20}{260} = -\frac{1}{13}, \qquad C \approx 94.41^\circ$$
2. The cosine is negative, so C is obtuse. The other two angles must be acute. For B, the angle between a and c:
$$\cos B = \frac{10^2 + 17^2 - 13^2}{2(10)(17)} = \frac{220}{340} = \frac{11}{17}, \qquad B \approx 49.68^\circ$$
3. Angle sum: $A \approx 180^\circ - 94.41^\circ - 49.68^\circ = 35.91^\circ$.

Answer: $A \approx 35.91^\circ$, $B \approx 49.68^\circ$, $C \approx 94.41^\circ$; the triangle is obtuse. **Check:** $17^2 = 289$ is greater than $10^2 + 13^2 = 269$, so the largest angle is obtuse.

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