Areas under a normal curve

When a measurement follows a normal model, what share of values lie in a range? And which value cuts off a given share?

WORDS TO KNOW

Cut: a value on the axis that splits the area under the curve.

Share: the fraction of all values in a range. Read as chance, it is the probability that one value picked at random lands there.

Expected count: a share times the number of items. It is what the model expects, not a promise.

$\Phi(z)$: the area under the standard normal curve to the left of z, read from a table or a calculator.

Percentile: the value with a given share of the area to its left. The 90th percentile has 0.90 to its left.

KEY IDEAS

1. From a value to a share

Draw the curve, mark the mean, and shade the region the question asks about. Turn each cut into a z-score; this is allowed because $X < a$ happens exactly when $Z < (a - \mu) \div \sigma$. Then read the area under $N(0, 1)$.

$$P(X < a) = \Phi\left(\frac{a - \mu}{\sigma}\right) \qquad P(X > a) = 1 - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

$$P(a < X < b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

For a negative z-score, use symmetry: $\Phi(-z) = 1 - \Phi(z)$. "Less than" and "at most" give the same share.

2. From a share back to a value

First turn the share into an area to the *left*: "the top 10%" has 0.90 to its left. Read the table backwards to find the z-score with that area (take the nearest entry). Then go z standard deviations from the mean.

$$x = \mu + z\sigma$$

For an area below 0.5, the z-score is negative, and the value is below the mean.

3. Working back to the model

The equation $x = \mu + z\sigma$ links four numbers. If a target fixes a value and its share, it fixes z, and then you can solve for the mean (given σ) or for σ (given the mean).

WATCH OUT

- **Sketch first.** The sketch tells you which rule to use and gives a size check: a region that is clearly less than half the curve cannot have area 0.8.
- **Standardize each model with its own mean and standard deviation.** The same cut can be a different number of standard deviations out in two models.
- **The model must fit.** A skewed variable, or one with a hard floor near its mean, gives wrong tail areas. Shares far out in the tails describe the model more than the world.
- **No share is ever exactly 0.** A normal curve puts some area beyond every cut, so a normal model can never promise that no value at all falls past a cut.

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WORKED EXAMPLES

Example 1. Apples from an orchard have masses modeled by $N(160, 20)$, in grams. What share weigh more than 196 g? In a crate of 600 apples, about how many is that?

1. Sketch: 196 is above the mean, so the region is a right tail, well under half the curve.
2. The cut as a z-score: $z = \dfrac{196 - 160}{20} = \dfrac{36}{20} = 1.8$.
3. A table or calculator gives $\Phi(1.8) \approx 0.9641$, so $P(X > 196) = 1 - 0.9641 = 0.0359$.
4. Count: 600 × 0.0359 = 21.54.

Answer: about 0.0359 of apples, so about 22 apples in the crate.

Example 2. For the same apples, $N(160, 20)$, what share weigh between 131 and 171 g?

1. Sketch: the strip holds the mean and reaches further left than right, so it should hold a good share, more than half.
2. Both cuts as z-scores: $\dfrac{131 - 160}{20} = -1.45$ and $\dfrac{171 - 160}{20} = 0.55$.
3. A table or calculator gives $\Phi(0.55) \approx 0.7088$ and $\Phi(1.45) \approx 0.9265$.
4. By symmetry, $\Phi(-1.45) \approx 1 - 0.9265 = 0.0735$.
5. Subtract the two left areas: 0.7088 − 0.0735 = 0.6353.

Answer: about 0.6353, a little under two thirds of the apples.

Example 3. For the same apples, find the mass that only the heaviest 14% exceed.

1. The heaviest 14% lie to the right of the cut, so 0.86 lies to its left.
2. Read the table backwards: $\Phi(1.08) \approx 0.8599$ is the nearest entry to 0.86, so $z \approx 1.08$.
3. Go 1.08 standard deviations above the mean: $x = 160 + 1.08(20) = 160 + 21.6 = 181.6$.

Answer: about 181.6 g. **Check:** going forward, $(181.6 - 160) \div 20 = 1.08$, and $1 - \Phi(1.08) \approx 0.14$.

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