The normal distribution

Many measurements pile up in a symmetric bell. What do the two numbers that fix a normal curve tell you, and how can you read them off its graph?

WORDS TO KNOW

Normal distribution: a model whose density curve is a symmetric bell with one peak.

Parameters: the two numbers μ and σ. Together they fix the whole curve.

Inflection point: a point where the curve changes from one way of bending to the other.

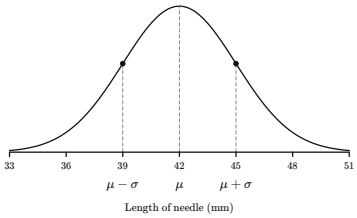
$N(\mu, \sigma)$**:** the normal distribution with mean μ (mu) and standard deviation σ (sigma).

Concave down / concave up: bending downward like a hilltop / bending upward like the bottom of a bowl.

KEY IDEAS

1. The mean is the center, the standard deviation is the bend

The peak stands above the mean μ. The curve is symmetric about it, so the mean, the median and the mode are all μ. The curve bends downward near the peak and upward further out, and it changes over exactly one standard deviation either side of the mean.



Peak above $x = \mu$, with height $\dfrac{1}{\sigma\sqrt{2\pi}} \approx \dfrac{0.399}{\sigma}$.

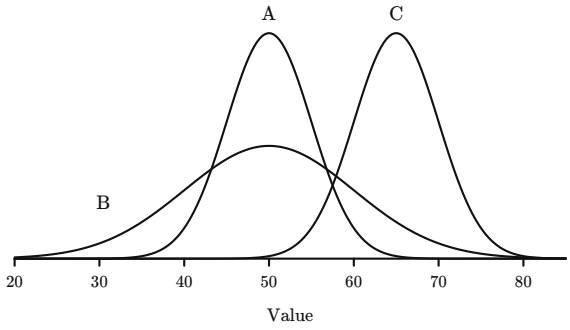
Inflection points above $\mu - \sigma$ and $\mu + \sigma$.

Draw the curve from $\mu - 3\sigma$ to $\mu + 3\sigma$. Past that it is almost on the axis, but it never touches it.

2. What each parameter does

Changing μ slides the curve left or right without changing its shape. Changing σ stretches it sideways. The area must stay 1, so a curve twice as wide is half as tall.

Same σ: same shape. Same μ: same center line. Smaller σ: taller, narrower bell.



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3. Symmetry and changing units

The share above $\mu + d$ equals the share below $\mu - d$, and half the area lies on each side of μ. If every value x is changed to $a + bx$ (with $b \neq 0$), the result is still normal.

$$N(\mu, \sigma) \longrightarrow N(a + b\mu,\ |b|\sigma)$$

Adding a moves the center only. Multiplying by b changes both the center and the spread.

WATCH OUT

- **The height is not a share.** Only area gives a share. With a small enough σ, the peak is taller than 1.
- **Check that a normal model fits.** It needs one symmetric hump. A skewed shape, two peaks, or values piled against a floor (such as 0) close to the mean do not fit. A mean far from the median is a warning sign.
- **Read which number is second.** Some books and calculators write $N(\mu, \sigma^2)$, with the variance second. There, $N(50, 36)$ means a standard deviation of 6.

WORKED EXAMPLES

Example 1. A normal curve has its peak above 9.5 and inflection points above 8.7 and 10.3. Write the model, find the peak height, and say over what range to draw it.

1. The peak stands above the mean, so $\mu = 9.5$.
2. Each inflection point is one standard deviation from the mean: 10.3 − 9.5 = 0.8, so $\sigma = 0.8$. The model is $N(9.5, 0.8)$.
3. The peak height is
$$\frac{1}{0.8\sqrt{2\pi}} \approx 0.499$$
4. Three standard deviations is 3 × 0.8 = 2.4. Draw from 9.5 − 2.4 = 7.1 to 9.5 + 2.4 = 11.9.

Answer: $N(9.5, 0.8)$, peak about 0.499, drawn from 7.1 to 11.9. **Check:** the mean is the midpoint of the inflection points, $(8.7 + 10.3) \div 2 = 9.5$.

Example 2. Job times for a plumber are modeled by $N(64, 10)$ in minutes. A job of t minutes costs $40 + 1.25t$ dollars. Give the model for the charges and its inflection points.

1. Changing every value to $40 + 1.25t$ keeps the curve normal.
2. New mean: 40 + 1.25 × 64 = 120. New standard deviation: 1.25 × 10 = 12.5. Adding 40 does not change the spread.
3. The inflection points are at 120 − 12.5 = 107.5 and 120 + 12.5 = 132.5.

Answer: $N(120, 12.5)$ in dollars, with inflection points above 107.5 and 132.5. **Check:** the job times at the old inflection points, 54 and 74 minutes, cost $40 + 1.25 \times 54 = 107.5$ and $40 + 1.25 \times 74 = 132.5$ dollars.

Example 3. A model $N(84, 5)$ gives a share of about 0.212 above 88. What share is below 80? Between 80 and 88? Between 84 and 88?

1. 88 is 4 above the mean and 80 is 4 below it. By symmetry the tail below 80 is the mirror image of the tail above 88: about 0.212.
2. The whole area is 1, so between 80 and 88 is 1 − 0.212 − 0.212 = 0.576.
3. Half the area is above 84, and 0.212 of it is above 88. So between 84 and 88 is 0.5 − 0.212 = 0.288.

Answer: about 0.212, 0.576 and 0.288. **Check:** 0.288 is half of 0.576, as symmetry says it must be.

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