Statistics Unit 1 quiz 5: The normal distribution, Normal calculations: areas and percentiles (answer key)

8 questions, 21 points

ANSWERS AT A GLANCE

1. $B = N(34, 3)$; its other inflection point is above 37.
2. about 0.3721
3. (a) about 0.0808 (b) about the 8th percentile; about 92% are slower.
4. (a) The ring road's, 5 times as tall. (b) Ring road: above 42 and 46 minutes; town route: above 28 and 48 minutes. (c) Each curve has total area 1, so the wider curve must be the lower one: the town route's curve is 5 times as wide and $\frac{1}{5}$ as tall.
5. (a) Inflection points above 0.5 and 3.5 minutes; drawn from −2.5 to 6.5 minutes. (b) Everything below 0: no time can be negative, and 0 is only about 1.3 standard deviations below the mean, so the curve puts a sizeable share of its area there. (c) The mean is above the median, which suggests a long right tail: the distribution is skewed to the right.
6. (a) $\mu = 400$ g, $\sigma = 8$ g (b) about 0.788 (c) about 0.394
7. (a) from about 289 to 331 ms (b) about 304 ms; about 40% of players react faster than this player, and about 60% slower.
8. (a) $\sigma \approx 2.5$ hours (b) about 0.0548 (c) about 10%, because 23.2 is as far above the mean as 16.8 is below it, and the curve is symmetric.

WORKED SOLUTIONS

1. Curve A is the normal curve $N(28, 3)$. Curve B is a normal curve with the same shape as A, and its left inflection point is directly above A's right inflection point. Write the model for B, and give the position of B's other inflection point. (2 points)

> **ANSWER**
> $B = N(34, 3)$; its other inflection point is above 37.
>
> **SOLUTION**
> A's inflection points are at 28 ± 3, so its right one is above 31. B has the same shape, so the same standard deviation, 3, and its left inflection point is one standard deviation below its mean: $\mu_B - 3 = 31$, so $\mu_B = 34$. The model is $N(34, 3)$, and its right inflection point is above 34 + 3 = 37.
>
> **SCORING**
> 1 point: $N(34, 3)$. 1 point: the other inflection point at 37.

2. The masses of the watermelons from one farm are modeled by $N(6{,}800, 1{,}200)$, in grams. Use the standard normal table or a calculator's normal function. What share of the watermelons weigh between 5,900 g and 7,100 g? Give it to four decimal places. (2 points)

> **ANSWER**
> about 0.3721
>
> **SOLUTION**
> The cuts as z-scores are $\frac{5{,}900 - 6{,}800}{1{,}200} = \frac{-900}{1{,}200} = -0.75$ and $\frac{7{,}100 - 6{,}800}{1{,}200} = \frac{300}{1{,}200} = 0.25$. The share between is the area left of the upper cut take away the area left of the lower one:
>
> $$\Phi(0.25) - \Phi(-0.75) \approx 0.5987 - 0.2266 = 0.3721.$$

3. The times of the members of a running club in a 5 km race are modeled by $N(25, 2)$, in minutes. Elena ran it in 22.2 minutes. Use the standard normal table or a calculator's normal function. (2 points)

(a) What share of the club's runners finish faster than Elena? Give it to four decimal places.

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(b) At about which percentile of the club's times is Elena's time, and about what percent of the club's runners are slower than she is?

ANSWER

(a) about 0.0808 (b) about the 8th percentile; about 92% are slower.

SOLUTION

(a) Faster means a shorter time. Elena's time as a z-score is $\frac{22.2 - 25}{2} = \frac{-2.8}{2} = -1.4$, so

$$P(X < 22.2) = \Phi(-1.4) = 1 - \Phi(1.4) \approx 1 - 0.9192 = 0.0808.$$

(b) About 8% of times lie below hers, so her time is at about the 8th percentile. The rest, about 1 − 0.0808 = 0.9192, or 92%, are slower. A low percentile is good news here, because lower times are faster.

4. A delivery driver can take the ring road or the town route to a warehouse. Trip times, in minutes, are modeled by $N(44, 2)$ on the ring road and $N(38, 10)$ on the town route. (3 points)

(a) Which route's curve has the taller peak, and how many times as tall is it?

(b) Give the inflection points of each curve.

(c) Another driver says, "The town route's curve is much wider, so its peak must be taller too, to hold all those extra times." Explain what is wrong.

ANSWER

(a) The ring road's, 5 times as tall. (b) Ring road: above 42 and 46 minutes; town route: above 28 and 48 minutes. (c) Each curve has total area 1, so the wider curve must be the lower one: the town route's curve is 5 times as wide and $\frac{1}{5}$ as tall.

SOLUTION

(a) The peak height is $\frac{1}{\sigma\sqrt{2\pi}}$, so the curve with the smaller standard deviation is taller. The ring road's $\sigma = 2$ is $\frac{1}{5}$ of the town route's 10, so its peak is 5 times as tall.

(b) At $\mu \pm \sigma$. Ring road: 44 − 2 = 42 and 44 + 2 = 46 minutes. Town route: 38 − 10 = 28 and 38 + 10 = 48 minutes.

(c) Both curves model every trip on their route, so each has total area exactly 1; the town route does not have "extra" trips. Its times are only more spread out. Stretching a curve to 5 times the width would multiply its area by 5, so to keep area 1 it must come down to $\frac{1}{5}$ of the height. The wider curve is the lower one.

SCORING

1 point: the ring road, 5 times. 1 point: all four inflection points. 1 point: total area 1 for each curve, so wider means lower.

5. A library's self-checkout machine records how long each customer takes, in minutes. The times have mean 2.0 and standard deviation 1.5. A librarian models them by $N(2.0, 1.5)$. (3 points)

(a) Where are the inflection points of the model's curve, and over what range would the curve be drawn, running three standard deviations either side of the mean?

(b) Which part of that range is impossible for a checkout time? Explain why this makes the normal model a poor choice.

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(c) The median time is 1.4 minutes. What does this, together with the mean, suggest about the shape of the distribution?

ANSWER

(a) Inflection points above 0.5 and 3.5 minutes; drawn from −2.5 to 6.5 minutes. (b) Everything below 0: no time can be negative, and 0 is only about 1.3 standard deviations below the mean, so the curve puts a sizeable share of its area there. (c) The mean is above the median, which suggests a long right tail: the distribution is skewed to the right.

SOLUTION

(a) The inflection points are at 2.0 − 1.5 = 0.5 and 2.0 + 1.5 = 3.5 minutes. Three standard deviations either side runs from 2.0 − 4.5 = −2.5 to 2.0 + 4.5 = 6.5 minutes.

(b) The part from −2.5 to 0 is impossible, since a time cannot be negative. 0 is only $\frac{2.0}{1.5} \approx 1.3$ standard deviations below the mean, between the left inflection point and $\mu - 3\sigma$, where the curve is still well above the axis, so the model gives a sizeable share of customers a negative time. Times piled against a floor of 0, with a standard deviation nearly as large as the mean, are usually skewed.

(c) On a normal curve the mean and median are equal. Here the mean, 2.0, is 0.6 minute above the median, 1.4: a few customers taking a long time pull the mean up. The distribution is skewed to the right.

SCORING

1 point: 0.5 and 3.5, and −2.5 to 6.5. 1 point: the part below 0 is impossible, and 0 is close enough to the mean that the curve has real area there. 1 point: mean above median, so skewed to the right.

6. The masses of the loaves from one bakery's oven, in grams, are modeled by a normal curve. The model gives a share of about 0.106 of loaves lighter than 390 g and the same share heavier than 410 g, and the curve's two inflection points are 16 g apart. (3 points)

(a) Find μ and σ.

(b) What share of loaves does the model give between 390 and 410 g?

(c) What share does it give between 400 and 410 g?

ANSWER

(a) $\mu = 400$ g, $\sigma = 8$ g (b) about 0.788 (c) about 0.394

SOLUTION

(a) A normal curve is symmetric about its mean, so two values with equal tails beyond them are mirror images in the mean: $\mu = \frac{390 + 410}{2} = 400$ g. The inflection points are at $\mu \pm \sigma$, so they are 2σ apart: $2\sigma = 16$ and $\sigma = 8$ g.

(b) The whole area is 1, so the share between 390 and 410 is what is left after both tails: 1 − 0.106 − 0.106 = 0.788.

(c) Half the area lies above the mean, 400, and 0.106 of it lies above 410, so the share between 400 and 410 is about 0.5 − 0.106 = 0.394 (or half of 0.788, by symmetry).

SCORING

1 point: $\mu = 400$ from the equal tails and $\sigma = 8$ from the inflection points. 1 point: 0.788 from 1 minus both tails. 1 point: 0.394.

7. A video game tests players' reaction times, which are modeled by $N(310, 25)$, in milliseconds. Use z-scores from the standard normal table (the nearest entry) and round times to the nearest whole millisecond. (3 points)

(a) Find the range that holds the middle 60% of reaction times.

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(b) Find the 40th percentile of the reaction times, and say what it means for a player whose time is exactly that.

ANSWER

(a) from about 289 to 331 ms (b) about 304 ms; about 40% of players react faster than this player, and about 60% slower.

SOLUTION

(a) The middle 60% leaves 20% in each tail, so the upper cut has 0.80 to its left: the nearest table entry is $\Phi(0.84) \approx 0.7995$, so $z \approx 0.84$, and the lower cut is $z \approx -0.84$. In milliseconds, $310 \pm 0.84(25) = 310 \pm 21$: from 289 to 331 ms.

(b) The 40th percentile has 0.40 to its left, below half, so its z-score is negative. By symmetry it is the negative of the cut with 0.60 to its left, and the nearest table entry is $\Phi(0.25) \approx 0.5987$, so $z \approx -0.25$. Then $x = 310 - 0.25(25) = 310 - 6.25 = 303.75$, about 304 ms. A player with that time is faster than about 60% of players: only about 40% have shorter times.

SCORING

(a) 2 points: 1 for $z \approx \pm 0.84$ from an area of 0.80 (or 0.20) to the left; 1 for both ends, 289 and 331 ms. (b) 1 point: 304 ms with a correct reading (about 40% of times are shorter). A student who reaches 316 ms by using $z = +0.25$ gets 0.

8. The time a rechargeable flashlight runs on one full charge is modeled as normal with mean 20 hours. Tests show that 10% of charges last less than 16.8 hours. Use z-scores from the standard normal table (the nearest entry). (3 points)

(a) Find the standard deviation of the run times.

(b) What share of charges last more than 24 hours? Give it to four decimal places.

(c) Without any more table work, what share of charges last more than 23.2 hours? Explain.

ANSWER

(a) $\sigma \approx 2.5$ hours (b) about 0.0548 (c) about 10%, because 23.2 is as far above the mean as 16.8 is below it, and the curve is symmetric.

SOLUTION

(a) 16.8 hours has 0.10 of run times to its left. The cut with 0.90 to its left is $z \approx 1.28$ (the nearest entry, $\Phi(1.28) \approx 0.8997$), so by symmetry 16.8 sits at $z \approx -1.28$. The equation $x = \mu + z\sigma$ gives

$$16.8 = 20 - 1.28\sigma, \qquad 1.28\sigma = 3.2, \qquad \sigma = 2.5 \text{ hours}.$$

(b) $z = \dfrac{24 - 20}{2.5} = 1.6$, so $P(X > 24) = 1 - \Phi(1.6) \approx 1 - 0.9452 = 0.0548$.

(c) 23.2 − 20 = 3.2 and 20 − 16.8 = 3.2, so 23.2 is the mirror image of 16.8 about the mean. The normal curve is symmetric about its mean, so the tail to the right of 23.2 has the same area as the tail to the left of 16.8: about 10% of charges.

SCORING

1 point each part. (a) 2.5 hours from $16.8 = 20 - 1.28\sigma$; a student who uses $z = +1.28$ gets a negative σ and 0 for the part. (b) 0.0548. (c) 10% with the symmetry reason; 10% with no reason gets 0.

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