Statistics Unit 1 worksheet: Normal calculations: areas and percentiles (answer key)

ANSWERS AT A GLANCE

1. (a) about 0.2266 (b) about 113 loaves
2. (a) $P(37 < X < 42)$ (b) about 0.6006 (c) about 0.0548
3. (a) about 0.1336 (b) about 334 dowels (c) about 0.1457 (accept 0.1458 from rounding each tail first); yes, the reject share rises by about 1.2 percentage points, from about 13.4% to 14.6%.
4. (a) under about 15.01 minutes (b) about 28.99 minutes
5. (a) about 51.1 minutes (b) a mean of about 43.4 minutes, about 2.6 minutes less than now
6. (a) It would need $300 = 299 - 2.33\sigma$, so $\sigma \approx -0.43$, which is impossible since $\sigma > 0$. With the mean below 300, more than half of all tiles are under 300 mm for every σ. (b) about 0.0082; yes, under 1% of tiles are short.

WORKED SOLUTIONS

1. A bakery's sourdough loaves have masses modeled by $N(820, 8)$, in grams. Use the standard normal table or a calculator's normal function, and give the share to four decimal places.

(a) What share of the loaves weigh less than 814 g?

(b) The bakery bakes 500 of these loaves in a month. About how many does the model expect to weigh less than 814 g? Round to the nearest whole loaf.

ANSWER

(a) about 0.2266 (b) about 113 loaves

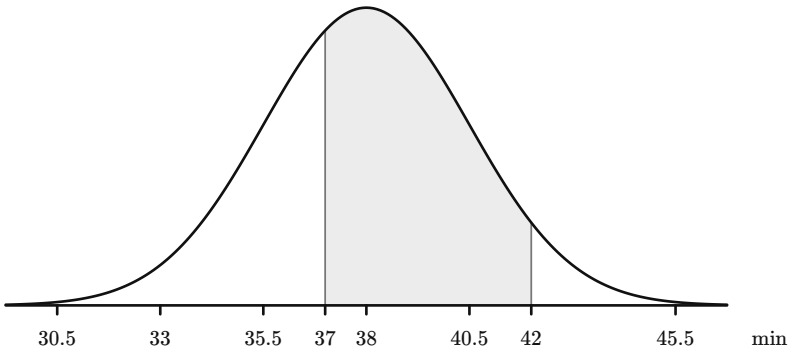
SOLUTION

(a) The cut as a z-score is $z = \dfrac{814 - 820}{8} = \dfrac{-6}{8} = -0.75$, so the region is a left tail. By symmetry,

$$P(X < 814) = \Phi(-0.75) = 1 - \Phi(0.75) \approx 0.2266.$$

(b) The model expects about 500 × 0.2266 of the loaves to be that light, which rounds to about 113 loaves. This is an expectation, not a promise: a real month might give a few more or a few fewer.

2. The times a ferry takes to cross a lake are modeled by $N(38, 2.5)$, in minutes. The figure shows the curve with one region shaded. Use a calculator's normal function and give shares to four decimal places.



(a) Write the shaded area as a probability about X, the time of one crossing.

(b) Find the shaded area.

(c) What share of crossings take longer than 42 minutes?

ANSWER

(a) $P(37 < X < 42)$ (b) about 0.6006 (c) about 0.0548

SOLUTION

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(a) The strip runs from 37 to 42 minutes, so its area is $P(37 < X < 42)$, the share of crossings that take between 37 and 42 minutes.

(b) The cuts as z-scores are $\dfrac{37-38}{2.5} = -0.4$ and $\dfrac{42-38}{2.5} = 1.6$. So

$$P(37 < X < 42) = \Phi(1.6) - \Phi(-0.4) \approx 0.9452 - 0.3446 = 0.6006.$$

The strip holds the center and more of the right side than the left, so a share a little over half fits the sketch.

(c) $P(X > 42) = 1 - \Phi(1.6) \approx 1 - 0.9452 = 0.0548$, about 5.5% of crossings.

3. A lathe turns wooden dowels whose diameters are modeled by $N(12.0, 0.2)$, in millimeters. A dowel is accepted only if its diameter is between 11.7 and 12.3 mm. Use a calculator's normal function and give shares to four decimal places.

(a) What share of dowels are rejected?

(b) About how many of a batch of 2,500 dowels does the model expect to be rejected?

(c) The lathe drifts so that its mean becomes 12.05 mm, with the same standard deviation. What share of dowels are rejected now? Did a drift of only 0.05 mm make a real difference?

ANSWER

(a) about 0.1336 (b) about 334 dowels (c) about 0.1457 (accept 0.1458 from rounding each tail first); yes, the reject share rises by about 1.2 percentage points, from about 13.4% to 14.6%.

SOLUTION

(a) The cuts are $\dfrac{11.7-12.0}{0.2} = -1.5$ and $\dfrac{12.3-12.0}{0.2} = 1.5$. A dowel is rejected if it falls in either tail. The tails are mirror images, so

$$P(\text{rejected}) = 2\big(1 - \Phi(1.5)\big) \approx 2(0.0668) = 0.1336.$$

(b) 2,500 × 0.1336 = 334, so about 334 dowels.

(c) Now the cuts are $\dfrac{11.7-12.05}{0.2} = -1.75$ and $\dfrac{12.3-12.05}{0.2} = 1.25$. The tails are no longer equal:

$$P(X < 11.7) = \Phi(-1.75) \approx 0.0401,$$
$$P(X > 12.3) = 1 - \Phi(1.25) \approx 0.1056,$$

so about 0.0401 + 0.1056 = 0.1457 are rejected. The left tail lost the strip from 1.5 to 1.75 standard deviations below the mean, but the right tail gained the strip from 1.25 to 1.5 above it, where the curve is higher, so it gained more than the left lost. The reject share went from about 13.4% to about 14.6%: about 30 more rejects in every 2,500 dowels.

4. A puzzle app times how long players take to finish its daily crossword. The times are modeled by $N(22, 3)$, in minutes. Use the z-score from the standard normal table (the nearest entry), and give times to two decimal places.

(a) The app gives a gold badge to the fastest 1% of players. What time must a player beat to earn the badge?

(b) What time do only the slowest 1% of players take longer than?

ANSWER

(a) under about 15.01 minutes (b) about 28.99 minutes

SOLUTION

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(a) The fastest players have the shortest times, so the badge cut has 1% of times to its left. The cut with 0.99 to its left is $z \approx 2.33$, so by symmetry the cut with 1% to its left is $z \approx -2.33$. Then

$$x = \mu + z\sigma = 22 - 2.33(3) = 22 - 6.99 = 15.01.$$

A player must finish in under about 15.01 minutes.

(b) The slowest 1% lie to the right of a cut with 0.99 to its left, $z \approx 2.33$: $x = 22 + 2.33(3) = 28.99$ minutes. The two cuts sit the same distance, 6.99 minutes, either side of the mean, because the curve is symmetric.

5. A bus company models the times its buses take to run one route by $N(46, 4)$, in minutes. Use the standard normal table or a calculator, and give times to one decimal place.

(a) The company wants to print a scheduled time that 90% of trips meet (take at most that long). What time should it print?

(b) The printed schedule says 50 minutes, and the company wants 95% of trips to meet it. If the standard deviation stays 4 minutes, what mean trip time is needed, and by how many minutes must the mean come down?

ANSWER

(a) about 51.1 minutes (b) a mean of about 43.4 minutes, about 2.6 minutes less than now

SOLUTION

(a) The scheduled time has 0.90 of trip times to its left, so $z \approx 1.28$. Then $x = 46 + 1.28(4) = 46 + 5.12 = 51.12$, about 51.1 minutes.

(b) Now 50 minutes must have 0.95 to its left, so it must sit at $z \approx 1.645$. The equation $x = \mu + z\sigma$ becomes

$$50 = \mu + 1.645(4) = \mu + 6.58, \qquad \mu = 43.42.$$

The mean must be about 43.4 minutes, which is 46 − 43.42 = 2.58, about 2.6 minutes faster than now.

6. A factory cuts floor tiles whose lengths are normal with mean 299 mm. A customer asks that 99% of tiles be at least 300 mm long.

(a) The manager says, "We will keep the mean at 299 mm and make the machine more precise." Use $z \approx -2.33$ for the 1st percentile to show that no standard deviation can meet the target, and explain why in words.

(b) The factory instead raises the mean to 300.6 mm and gets the standard deviation down to 0.25 mm. Use a calculator's normal function to find the share of tiles shorter than 300 mm, to four decimal places. Is the customer's target met?

ANSWER

(a) It would need $300 = 299 - 2.33\sigma$, so $\sigma \approx -0.43$, which is impossible since $\sigma > 0$. With the mean below 300, more than half of all tiles are under 300 mm for every σ. (b) about 0.0082; yes, under 1% of tiles are short.

SOLUTION

(a) For 99% to be at least 300 mm, 300 must be the 1st percentile, at $z \approx -2.33$. Then $x = \mu + z\sigma$ gives

$$300 = 299 - 2.33\sigma, \qquad \sigma = \frac{299 - 300}{2.33} \approx -0.43.$$

A standard deviation is never negative, so no machine can do it. In words: the mean 299 is the median of a normal model, so half the tiles are under 299 mm and so under 300 mm, however small the spread. A

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tighter machine only packs more tiles close to 299, which is still too short.

(b) The cut as a z-score is $\frac{300 - 300.6}{0.25} = -2.4$, so $P(X < 300) = \Phi(-2.4) \approx 0.0082$. That is about 0.82%, under the 1% the customer allows, so the target is met.

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