Statistics Unit 1 worksheet: The normal distribution (answer key)

ANSWERS AT A GLANCE

1. (a) $N(34, 2)$ (b) above 32 and 36 cm (c) above 30 and 38 cm; the reader's peak would be $\frac{1}{2}$ as tall as the true peak.
2. (a) about 1.60 per cm (b) The height of a density curve is a density, a share per centimeter, not a share; only an area gives a share. (c) $\frac{1}{2}$
3. (a) $N(45, 6)$ (b) above 39 and 51 kg (c) The charge curve's peak is $\frac{1}{4}$ as tall.
4. (a) No: 0 books is only one standard deviation below the mean, so the curve would put a sizeable share of its area on negative counts. (b) No: 20 is only one standard deviation above the mean, so the curve would put a sizeable share of scores above the top mark. (c) No: two groups make two humps. (d) Yes: one symmetric hump, and the impossible lengths are 14 standard deviations below the mean, where the curve is all but on the axis.
5. (a) $\mu = 110$ cm, $\sigma = 12$ cm; the inflection points are above 98 and 122 cm. (b) about 0.0332 per cm (c) about 0.252; about 0.252; about 0.496
6. (a) Curve 2: its peak is above 14, and it is the narrower, taller curve, as $\sigma = 1.5$ is the smaller standard deviation. (b) P: above 12.5 and 15.5 cm; Q: above 14 and 20 cm. Q's left inflection point, at 14 cm, is below P's peak. (c) P: about 0.266 per cm; Q: about 0.133 per cm. P's peak is 2 times as tall, because P's curve is half as wide and its area must still be 1.

WORKED SOLUTIONS

1. A spreadsheet guide gives the model for the lengths of cucumbers from one greenhouse, in centimeters, as $N(34, 4)$, with a note that it writes the variance second.

(a) Write the model in this lesson's notation, $N(\mu, \sigma)$.

(b) Where are the inflection points of the curve?

(c) A reader misses the note and takes 4 as the standard deviation. Where would the inflection points of the reader's curve be, and how would its peak height compare with the true curve's?

> **ANSWER**
>
> (a) $N(34, 2)$ (b) above 32 and 36 cm (c) above 30 and 38 cm; the reader's peak would be $\frac{1}{2}$ as tall as the true peak.
>
> **SOLUTION**
>
> (a) The variance is 4, so the standard deviation is $\sigma = \sqrt{4} = 2$ cm, and the model is $N(34, 2)$.
>
> (b) The inflection points are one standard deviation either side of the mean: 34 − 2 = 32 cm and 34 + 2 = 36 cm.
>
> (c) The reader's curve has $\sigma = 4$, so its inflection points are at 34 − 4 = 30 and 34 + 4 = 38 cm. The peak height is $\frac{1}{\sigma\sqrt{2\pi}}$, so a standard deviation 2 times as large gives a peak $\frac{1}{2}$ as tall: the reader's curve is 2 times wider and 2 times lower.

2. The diameters of buttons from one machine are modeled by $N(4.0, 0.25)$, in centimeters.

(a) Find the height of the peak of the curve, to three significant figures.

(b) A student says, "So the model says 160% of the buttons are 4.0 cm across." Explain the mistake.

(c) What share of buttons does the model give with diameters less than 4.0 cm?

> **ANSWER**
>
> (a) about 1.60 per cm (b) The height of a density curve is a density, a share per centimeter, not a share; only an area gives a share. (c) $\frac{1}{2}$

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SOLUTION

(a) The peak height is $\frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{0.25\sqrt{2\pi}} \approx 1.60$ per centimeter.

(b) A share of the buttons is an area under the curve, never its height. The height 1.60 is a share per centimeter: it says that a narrow strip of diameters near 4.0 cm, say 0.01 cm wide, holds about $1.60 \times 0.01 = 0.016$ of the buttons. The curve can be taller than 1 because σ is small, yet the total area under it is still 1. And the share at exactly 4.0 cm, an interval of width 0, is 0.

(c) A normal curve is symmetric about its mean, 4.0 cm, so half the area lies on each side: $\frac{1}{2}$ of the buttons.

3. A firewood seller charges a delivery fee of 19 dollars plus 4 dollars per kilogram of wood. The charges for her deliveries are modeled by $N(199, 24)$, in dollars.

(a) Find the model for the masses of wood in her deliveries, in kilograms.

(b) Where are the inflection points of the curve for the masses?

(c) How does the peak height of the curve for the charges compare with that of the curve for the masses?

ANSWER

(a) $N(45, 6)$ (b) above 39 and 51 kg (c) The charge curve's peak is $\frac{1}{4}$ as tall.

SOLUTION

(a) A delivery of x kg costs $19 + 4x$ dollars. If the masses are $N(\mu, \sigma)$, the charges are $N(19 + 4\mu,\ 4\sigma)$. So $19 + 4\mu = 199$, giving $4\mu = 199 - 19$ and $\mu = 45$ kg, and $4\sigma = 24$, giving $\sigma = 6$ kg. The model is $N(45, 6)$.

(b) One standard deviation either side of the mean: 45 − 6 = 39 kg and 45 + 6 = 51 kg.

(c) The peak height is $\frac{1}{\sigma\sqrt{2\pi}}$. The charges have a standard deviation 4 times as large (24 against 6), so their peak is $\frac{1}{4}$ as tall. Each dollar of charge covers only $\frac{1}{4}$ kg of wood, so the share per dollar is $\frac{1}{4}$ of the share per kilogram.

4. Decide whether a normal model is reasonable for each variable, and explain.

(a) The number of books each student at a school borrowed from the library last month: mean 1.5, standard deviation 1.5.

(b) Scores on a vocabulary quiz marked out of 20: mean 18, standard deviation 2.

(c) Finishing times at a 5 km charity event in which about half the entrants ran and half walked.

(d) Lengths of adult salmon caught in one river: mean 70 cm, standard deviation 5 cm, with a histogram that is a single symmetric hump.

ANSWER

(a) No: 0 books is only one standard deviation below the mean, so the curve would put a sizeable share of its area on negative counts. (b) No: 20 is only one standard deviation above the mean, so the curve would put a sizeable share of scores above the top mark. (c) No: two groups make two humps. (d) Yes: one symmetric hump, and the impossible lengths are 14 standard deviations below the mean, where the curve is all but on the axis.

SOLUTION

(a) Not reasonable. A count of books cannot be negative, but 1.5 − 1.5 = 0 is only one standard deviation below the mean, inside the range where the curve is still well above the axis. The model would give a sizeable share of students fewer than 0 books. Counts piled against a floor of 0, with a standard deviation as large as the mean, are skewed to the right (and they are whole numbers, too).

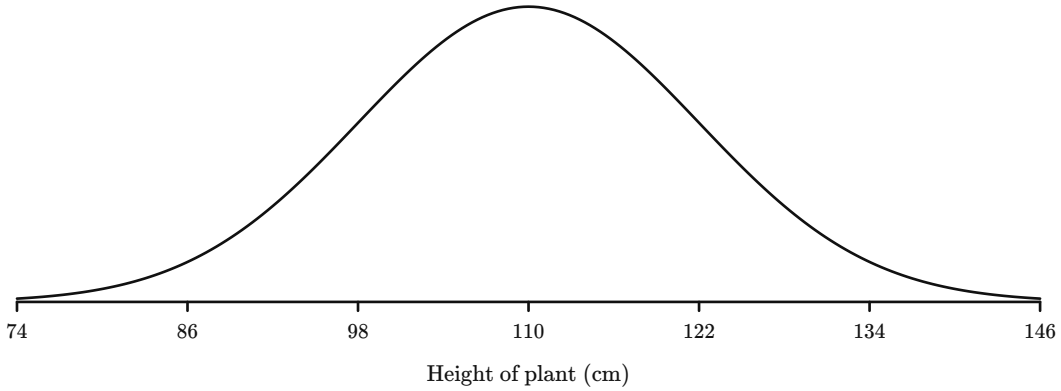
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(b) Not reasonable. No score can be above 20, but 18 + 2 = 20 is only one standard deviation above the mean, so the model would give a sizeable share of scores above the top mark. Scores piled against a ceiling are skewed to the left.

(c) Not reasonable. Runners and walkers take very different times, so the histogram would have two humps, one for each group, and a normal curve has only one.

(d) Reasonable. The histogram has the single symmetric hump of a normal curve. A length below 0 is impossible, but 0 is $\frac{70}{5} = 14$ standard deviations below the mean, far past $\mu - 3\sigma = 55$ cm, beyond which the curve can hardly be told apart from the axis. The area below 0 is far too small to matter.

5. The figure shows the normal model for the heights of sunflower plants eight weeks after sowing, in centimeters. The curve is drawn from three standard deviations below the mean to three standard deviations above it, and the tick marks are equally spaced.



(a) Read μ and σ from the figure, saying which features you used. Above which heights are the inflection points?

(b) Find the height of the peak, to three significant figures, with its unit.

(c) The model gives a share of about 0.248 of plants between 110 and 118 cm tall. What share does it give taller than 118 cm? Shorter than 102 cm? Between 102 and 118 cm?

ANSWER

(a) $\mu = 110$ cm, $\sigma = 12$ cm; the inflection points are above 98 and 122 cm. (b) about 0.0332 per cm (c) about 0.252; about 0.252; about 0.496

SOLUTION

(a) The peak stands above the mean, so $\mu = 110$ cm. The curve runs from 74 to 146, which is 6 standard deviations: $146 - 74 = 72 = 6\sigma$, so $\sigma = 12$ cm (the seven ticks are one standard deviation apart). The inflection points are one standard deviation either side of the mean, above 110 − 12 = 98 and 110 + 12 = 122 cm.

(b) The peak height is $\frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{12\sqrt{2\pi}} \approx 0.0332$ per centimeter. It is a density, a share per centimeter, not a share.

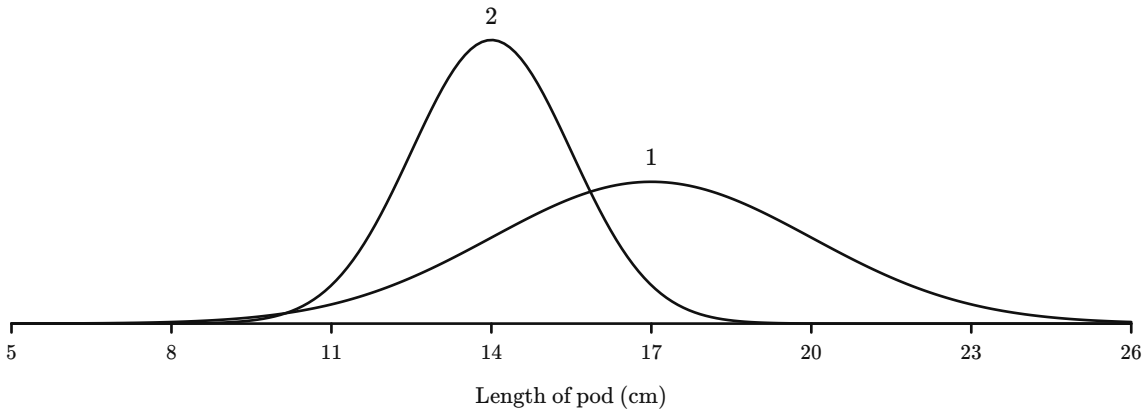
(c) Half the area lies above the mean, 110, and 0.248 of it lies between 110 and 118, so about 0.5 − 0.248 = 0.252 lies above 118.

102 is 8 below the mean and 118 is 8 above it. The curve is symmetric about 110, so the share below 102 is the mirror image of the share above 118: about 0.252.

The share between 102 and 118 is the two mirror-image pieces from 102 to 110 and from 110 to 118: about 2 × 0.248 = 0.496 (or 1 − 0.252 − 0.252).

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6. Two varieties of bean are grown side by side. The lengths of their pods, in centimeters, are modeled by $P = N(14, 1.5)$ and $Q = N(17, 3)$. The figure shows both curves, labeled 1 and 2.



(a) Which curve is P? Give two features of the curves that show it.

(b) Give the inflection points of each curve. One of them sits directly below the other curve's peak: which one?

(c) Find the peak height of each curve, to three significant figures. How many times as tall as Q's peak is P's, and why must it be?

ANSWER

(a) Curve 2: its peak is above 14, and it is the narrower, taller curve, as $\sigma = 1.5$ is the smaller standard deviation. (b) P: above 12.5 and 15.5 cm; Q: above 14 and 20 cm. Q's left inflection point, at 14 cm, is below P's peak. (c) P: about 0.266 per cm; Q: about 0.133 per cm. P's peak is 2 times as tall, because P's curve is half as wide and its area must still be 1.

SOLUTION

(a) A normal curve's peak stands above its mean, and curve 2 peaks above 14, curve 1 above 17. Also, a smaller standard deviation gives a narrower, taller curve: P's $\sigma = 1.5$ is half of Q's $\sigma = 3$, and curve 2 is the narrow one. So curve 2 is P.

(b) The inflection points are at $\mu \pm \sigma$. P: 14 − 1.5 = 12.5 and 14 + 1.5 = 15.5 cm. Q: 17 − 3 = 14 and 17 + 3 = 20 cm. Q's left inflection point, 14 cm, is exactly P's mean, where P's peak stands.

(c) The peak height is $\frac{1}{\sigma\sqrt{2\pi}}$. P: $\frac{1}{1.5\sqrt{2\pi}} \approx 0.266$ per cm. Q: $\frac{1}{3\sqrt{2\pi}} \approx 0.133$ per cm. P's peak is 2 times as tall. Q's standard deviation is twice P's, so its curve is stretched to twice the width; each curve has total area 1, so the twice-as-wide curve must be half as tall.

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