Statistics Unit 1 worksheet: z-scores and the empirical rule (answer key)

ANSWERS AT A GLANCE

1. (a) $z = 1.75$ (b) $x = 73$ (c) $z = 2.5$ (d) $x = 7.5$ (e) $z = -2$ (f) $x = 2{,}610$
2. (a) Priya: chemistry $z = 1.5$, history $z = 1.8$. Diego: chemistry $z = -0.75$, history $z = -0.8$. (b) Priya did better in history; Diego did better in chemistry. (c) 61.6
3. (a) Monday −1.94, not unusual; Tuesday 2.23, unusual; Thursday 0.29, not unusual; Saturday −2.14, unusual. (b) From 350 to 490 visitors.
4. (a) $z = -2$, $z = -1$, $z = 2$, $z = 3$ (b) about 97.35% (c) about 47.5% (d) about 420 lambs
5. (a) $z = -1.5$ and $z = 2.5$ (b) $\mu = 30$ minutes, $\sigma = 8$ minutes (c) 54 minutes
6. (a) $z = 1.75$: the longest flight is 1.75 standard deviations above the mean distance. (b) Andre gives the distance from the mean, not the number of standard deviations, and a z-score has no units; Mei divides by the variance instead of the standard deviation; Ryan reads standard deviations as centimeters, and that flight is 6 cm shorter, 174 cm long.

WORKED SOLUTIONS

1. Find the z-score z or the value x, as each part asks.

(a) z when $x = 74,\ \mu = 60,\ \sigma = 8$

(b) x when $z = -1.4,\ \mu = 80,\ \sigma = 5$

(c) z when $x = 6.7,\ \bar{x} = 4.7,\ s = 0.8$

(d) x when $z = 2.6,\ \mu = 6.2,\ \sigma = 0.5$

(e) z when $x = 0.88,\ \mu = 1,\ \sigma = 0.06$

(f) x when $z = 0.6,\ \mu = 2{,}400,\ \sigma = 350$

ANSWER

(a) $z = 1.75$ (b) $x = 73$ (c) $z = 2.5$ (d) $x = 7.5$ (e) $z = -2$ (f) $x = 2{,}610$

SOLUTION

A z-score is the distance from the mean divided by the standard deviation, and a value with z-score z is $x = \mu + z\sigma$.

(a) $\dfrac{74 - 60}{8} = \dfrac{14}{8} = 1.75$

(b) $80 + (-1.4)(5) = 80 - 7 = 73$

(c) $\dfrac{6.7 - 4.7}{0.8} = \dfrac{2}{0.8} = 2.5$

(d) $6.2 + (2.6)(0.5) = 6.2 + 1.3 = 7.5$

(e) $\dfrac{0.88 - 1}{0.06} = \dfrac{-0.12}{0.06} = -2$

(f) $2{,}400 + (0.6)(350) = 2{,}400 + 210 = 2{,}610$

2. Priya and Diego sat the same chemistry and history tests as the rest of their class. The chemistry scores had mean 68 and standard deviation 8; the history scores had mean 74 and standard deviation 5.

	Chemistry	History
Priya	80	83
Diego	62	70

(a) Find all four z-scores.

(b) For each student, on which test did they do better compared with the class?

(c) What chemistry score would have put Diego in the same place among the class as his history score did?

ANSWER

(a) Priya: chemistry $z = 1.5$, history $z = 1.8$. Diego: chemistry $z = -0.75$, history $z = -0.8$. (b) Priya did better in history; Diego did better in chemistry. (c) 61.6

SOLUTION

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(a) Priya: $\dfrac{80-68}{8}=\dfrac{12}{8}=1.5$ and $\dfrac{83-74}{5}=\dfrac{9}{5}=1.8$.
Diego: $\dfrac{62-68}{8}=\dfrac{-6}{8}=-0.75$ and $\dfrac{70-74}{5}=\dfrac{-4}{5}=-0.8$.
(b) Priya is 1.8 standard deviations above the mean in history and 1.5 in chemistry, so history was her better test, even though she beat the chemistry mean by more points.
Diego is below the mean on both. In chemistry he is 0.75 standard deviations below and in history 0.8 below, so chemistry was his better test, even though he was 6 points below the mean there and only 4 in history. A history standard deviation is only 5 points, so 4 points there counts for more.
(c) The same place means the same z-score, -0.8, on the chemistry scale: $x=68+(-0.8)(8)=68-6.4=61.6$.

3. A public library counted its visitors on a sample of days. The daily counts have mean $\bar{x}=420$ and standard deviation $s=35$. Call a day unusual when its z-score is greater than 2 or less than -2.

Day	Monday	Tuesday	Thursday	Saturday
Visitors	352	498	430	345

(a) Find the z-score of each day, to two decimal places, and say whether the day is unusual.
(b) From what count to what count is a day's number of visitors not unusual?

ANSWER
(a) Monday -1.94, not unusual; Tuesday 2.23, unusual; Thursday 0.29, not unusual; Saturday -2.14, unusual. (b) From 350 to 490 visitors.

SOLUTION
(a) Monday: $\dfrac{352-420}{35}=\dfrac{-68}{35}\approx -1.94$, not unusual.
Tuesday: $\dfrac{498-420}{35}=\dfrac{78}{35}\approx 2.23$, unusual.
Thursday: $\dfrac{430-420}{35}=\dfrac{10}{35}\approx 0.29$, not unusual.
Saturday: $\dfrac{345-420}{35}=\dfrac{-75}{35}\approx -2.14$, unusual.
Monday and Saturday are only 7 visitors apart, but the line at $z=-2$ falls between them.
(b) A day is not unusual when $-2\le z\le 2$, that is, within two standard deviations of the mean: from 420 − 2(35) = 350 to 420 + 2(35) = 490 visitors.

4. A sheep farm weighs every lamb at birth. The birth masses are modeled by $N(4.6,0.5)$, in kilograms. Use the empirical rule: for a normal model, about 68%, 95% and 99.7% of values lie within one, two and three standard deviations of the mean.

(a) Find the z-scores of birth masses of 3.6, 4.1, 5.6 and 6.1 kg.
(b) Estimate the share of lambs born between 3.6 and 6.1 kg.
(c) Estimate the share of lambs born between 4.6 and 5.6 kg.
(d) The farm expects 500 lambs this spring. About how many will weigh more than 4.1 kg at birth?

ANSWER
(a) $z=-2$, $z=-1$, $z=2$, $z=3$ (b) about 97.35% (c) about 47.5% (d) about 420 lambs

SOLUTION
(a) $\dfrac{3.6-4.6}{0.5}=\dfrac{-1}{0.5}=-2$, $\dfrac{4.1-4.6}{0.5}=-1$, $\dfrac{5.6-4.6}{0.5}=2$ and $\dfrac{6.1-4.6}{0.5}=3$.

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(b) From $z = -2$ to $z = 3$. The part from −2 to 0 is half of 95%, which is 47.5%, and the part from 0 to 3 is half of 99.7%, which is 49.85%. Together that is about 97.35%, or about 97%.

(c) From $z = 0$ to $z = 2$: half of 95%, about 47.5%.

(d) The share below $z = -1$ is half of 100% − 68% = 32%, which is 16%, so about 84% of lambs weigh more than 4.1 kg. That is 0.84 × 500 = 420 lambs.

5. A pizza shop times its deliveries, from the order to the door. A delivery of 18 minutes is one and a half standard deviations shorter than the mean delivery time, and a delivery of 50 minutes is two and a half standard deviations longer than the mean.

(a) Write the z-score of each of the two deliveries.

(b) Find the mean and the standard deviation of the delivery times.

(c) How long is a delivery that is three standard deviations longer than the mean?

ANSWER

(a) $z = -1.5$ and $z = 2.5$ (b) $\mu = 30$ minutes, $\sigma = 8$ minutes (c) 54 minutes

SOLUTION

(a) A z-score counts standard deviations from the mean, negative below it and positive above: 18 minutes has $z = -1.5$ and 50 minutes has $z = 2.5$.

(b) Each z-score gives an equation through $x = \mu + z\sigma$:

$$18 = \mu - 1.5\sigma,$$
$$50 = \mu + 2.5\sigma.$$

Taking the first equation from the second removes μ: $32 = (2.5 + 1.5)\sigma$, so $\sigma = 8$ minutes. Then $\mu = 18 + 1.5(8) = 30$ minutes. As a check, $\dfrac{50 - 30}{8} = 2.5$.

(c) $x = \mu + 3\sigma = 30 + 3(8) = 54$ minutes.

6. Leah's class flew paper airplanes. The flight distances had mean 180 cm and standard deviation 12 cm, and the longest flight was 201 cm.

(a) Find the z-score of the longest flight and say what it means.

(b) Three classmates made the claims below. Explain what is wrong with each, and give the correct statement.

Andre: "The z-score is 21 cm, because the flight is 21 cm longer than the mean."

Mei: "The variance is 144, so the z-score is $\dfrac{21}{144} \approx 0.15$."

Ryan: "A flight with z-score −0.5 is 0.5 cm shorter than the mean."

ANSWER

(a) $z = 1.75$: the longest flight is 1.75 standard deviations above the mean distance. (b) Andre gives the distance from the mean, not the number of standard deviations, and a z-score has no units; Mei divides by the variance instead of the standard deviation; Ryan reads standard deviations as centimeters, and that flight is 6 cm shorter, 174 cm long.

SOLUTION

(a) $z = \dfrac{201 - 180}{12} = \dfrac{21}{12} = 1.75$. The longest flight is 1.75 standard deviations longer than the class's mean flight.

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(b) Andre: 21 cm is the distance from the mean. A z-score divides that distance by the standard deviation, $\frac{21}{12} = 1.75$, and since both are in centimeters the units cancel. The z-score is 1.75, with no units.

Mei: the bottom of a z-score is the standard deviation, 12 cm, not the variance 144. The variance is in square centimeters, so $\frac{21}{144}$ is not even free of units. The z-score is $\frac{21}{12} = 1.75$.

Ryan: a z-score counts standard deviations, not centimeters. A z-score of -0.5 is half a standard deviation below the mean, $0.5 \times 12 = 6$ cm shorter, so that flight is $180 - 6 = 174$ cm long.

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